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# On the degree of ambiguity of finite automata

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*Abstract*

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We investigate the ambiguity behavior of finite automata in connection with their inner structure. We show that the degree of ambiguity of a finitely ambiguous nondeterministic finite automaton (NFA) with $n$ states is at most $5^{n/2} \cdot n^n$. There is a simple criterion which characterizes the infinite degree of ambiguity of an NFA, and which is decidable in polynomial time. The degree of growth of the ambiguity of an NFA is computable in polynomial time. Starting from the first result, we discuss the maximal finite degree of ambiguity of an NFA with $n$ states, and we present subclasses of NFAs where this quantity is of order $2^{\Theta(n)}$.

## 0. Introduction

The degree of ambiguity is a structural parameter of a finite automaton. Let $x$ be an input word of a nondeterministic finite automaton (NFA) $M$. The degree of ambiguity of $x$ in $M$ ($\mathrm{da}_M(x)$) is defined as the number of all accepting paths for $x$. The degree of ambiguity of $M$ is the maximal degree of ambiguity of an input word of $M$ or is infinite, depending on whether or not a maximum exists. In the former (latter) case $M$ is called finitely (infinitely) ambiguous. The degree of growth of the ambiguity of $M$ is defined as the minimal degree of a polynomial $h$ over $\mathbb{N}_0$ ($\mathbb{N}_0$ denotes the semiring of all nonnegative integers) such that for each input word $x$ of $M$ $\mathrm{da}_M(x)$ is at most $h(|x|)$ if such a polynomial exists, or is infinite otherwise. In the latter case $M$ is called exponentially ambiguous. We will abbreviate the degree of growth of the

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ambiguity of $M$ by "*degree* of $M$" and ask the reader not to mix up this notion with the "degree of *ambiguity*".

Only recently, the degree of ambiguity received attention in connection with the equivalence problems for NFAs and finite tree automata, and with the succinctness of representation of regular languages. For the latter we refer to [19] and [8]. For the former we mention that the equivalence problem is PSPACE-complete for NFAs (see [4]) and DEXPTIME-complete (w.r.t. log-space reductions) for finite tree automata [17]. For any fixed integrer $k$, however, the equivalence of NFAs with degree of ambiguity at most $k$ can be tested in polynomial time [19, 11]. The same assertion holds true even for finite tree automata [17].

The following fundamental result was proved independently by Mandel and Simon [13], Jacob [9] and Reutenauer [16]: It is decidable whether or not an NFA is finitely ambiguous. In fact, the aim of the three mentioned papers was to show: (a) It is decidable whether or not a finitely generated monoid of matrices with entries in $\mathbb{N}_0$ (or in a larger semiring) is finite [13, 9]. (b) There is an algorithm which computes the degree of an NFA [16]. Moreover, from this work the following upper bounds can be derived. The degree of ambiguity of a finitely ambiguous NFA with $n$ states and input alphabet $\Sigma$ is at most

- $n^{2^{3^{3 \cdot n^2+1}}}$ [13],
- $n^{f(n, \# \Sigma)}$ where $f$ is a recursive function [9],
- $2^n \cdot n^{2n} \cdot 2^{4 \cdot n^3}$ [16].

The following results arose independently of [13, 9, 16]. Given any fixed integer $k$, it can be tested in polynomial time whether or not the degree of ambiguity of an NFA is greater than $k$ (Stearns and Hunt III [19], see also [24]). Applying a nice matrix algorithm, it can be decided in polynomial space whether or not an NFA is finitely ambiguous (Chan and Ibarra [3]). The problem "decide on input of an NFA $M$ and of $i \in \mathbb{N}_0$ whether or not the degree of ambiguity of $M$ is greater than $i$" is PSPACE-complete [3].

In this paper we clearly follow the contents of [22] and of the first five chapters in [20]. Our main results are:

(1) The degree of ambiguity of a finitely ambiguous NFA with $n$ states is at most $5^{n/2} \cdot n^n$ (see Section 2).

(2) There is a simple criterion (IDA) which characterizes the infinite degree of ambiguity of an NFA, and which is decidable in polynomial time (see Section 3).

Generalizing (2), we obtain a polynomial-time algorithm which computes the degree of an NFA (see Section 4). In fact, this algorithm is based on the criteria (EDA) and ($\text{IDA}_d$) which characterize the properties "$M$ is exponentially ambiguous" and "the degree of $M$ is at least $d$", respectively, of an NFA $M$ ($d \in \mathbb{N}_0$).

In Section 5 we discuss, starting from (1), the maximal finite degree of ambiguity of an NFA with $n$ states. We present subclasses of NFAs restricted to which this quantity is a function of order $2^{\Theta(n)}$.

By reduction, (1) and (2) can be generalized to NFAs with $\varepsilon$-moves [20, Chapter 6]. Moreover, all our results can be automatically transformed into assertions on finitely generated monoids of matrices with entries in $\mathbb{N}_0$ (see [20, Chapter 7] and [11, 23]). In order to prove (1) and (2) we introduce new elementary methods. First of all, we show that it is sufficient to consider chain NFAs, which have a restricted structure. Then, for every input word $x$ we investigate a graph which describes all accepting paths for $x$ in the NFA, and we use "pumping arguments" in these graphs. The criterion (IDA) describes a simple reason for an NFA to be infinitely ambiguous. In fact, we show that this is the only reason.

From another point of view, our proof of the characterization part of (2) reads as follows. At first, reduce the result to the case that the NFA in question has only one input symbol, and then verify the correctness in that easy particular case (see also [20, Theorem 5.1]). As pointed out by the referee, this basic frame was already used in the above-mentioned paper by Mandel and Simon [13]. Indeed, from [13] a proof of the correctness of the criterion (IDA) can be easily derived. In order to carry out the reduction to the case of one input symbol, [13] offers to use a theorem by McNaughton and Zalcstein [14] restricted to the nonnegative integers and also contains a combinatorial proof of that restricted form of the theorem. Our methods contribute a *short* combinatorial proof of the above reduction and, thus, a new correctness proof of the new criterion (IDA).

Independently of us, Ibarra and Ravikumar [7] exhibited a criterion which is equivalent to (IDA), and which can be tested in double exponential time. The above-mentioned criterion (EDA) was already used in [16, 7].

Only recently, the results and techniques presented in this paper turned out to be stimulating for further research on finite automata. Using an estimate of Baron [2], the upper bound in (1) can be improved to $2^{1+k_2 \cdot n} \cdot n^n$, where $k_2 < 0.7956$ (note that $5^{n/2} = 2^{k_1 \cdot n}$, where $k_1 \approx 1.1610$). In [11] this improvement and some of our results and proofs are presented in the context of the theory of formal power series. A new topological approach exhibited by Leung [12] yields an alternative proof of the characterization part of (2). In [23] the authors present a "nonramification" lemma for NFAs and apply it to finitely generated monoids of matrices with entries in $\mathbb{N}_0$. In fact, this lemma allows to shorten an input word of a finitely ambiguous NFA without changing its ambiguity behavior. Extending these ideas and the techniques presented in this paper the second author generalizes (1) and (2) to finite tree automata [18].

## 1. Definitions and notations

A *nondeterministic finite automaton* (NFA) is a 5-tuple $M = (Q, \Sigma, \delta, Q_I, Q_F)$ where $Q$ and $\Sigma$ denote nonempty, finite sets of states and input symbols, respectively, $Q_I, Q_F \subseteq Q$ denote sets of initial and final (or accepting) states, respectively, and $\delta$ is a subset of $Q \times \Sigma \times Q$. $\Sigma$ is called the input alphabet of $M$, $\delta$ is called the transition relation of $M$. Each element of $\delta$ denotes a *transition* of $M$.

The mode of operation of $M$ is described by paths. A *path* $\pi$ (of length $m$) for $x$ in $M$ leading from $p$ to $q$ is a word $(q_1,x_1)\ldots(q_m,x_m)q_{m+1}\in(Q\times\Sigma)^m\cdot Q$ so that $(q_1,x_1,q_2),\ldots,(q_m,x_m,q_{m+1})$ are transitions of $M$ and the equalities $x=x_1\ldots x_m, p=q_1$ and $q=q_{m+1}$ hold. $\pi$ is said to consume $x$. $\pi$ is called *accepting* if $p\in Q_{\mathrm{I}}$ and $q\in Q_{\mathrm{F}}$. The *language recognized* by $M$, denoted by $L(M)$, is the set of words consumed by all accepting paths in $M$.

For each $(p,x,q)\in Q\times\Sigma^*\times Q$ $\mathrm{da}_M(p,x,q)$ is defined as the number of all paths for $x$ in $M$ leading from $p$ to $q$. We define $\hat{\delta}:=\{(p,x,q)\in Q\times\Sigma^*\times Q\,|\,\mathrm{da}_M(p,x,q)\neq 0\}$. Note that $\delta=\hat{\delta}\cap Q\times\Sigma\times Q$. We rename $\hat{\delta}$ by $\delta$.

The *degree of ambiguity* of $x\in\Sigma^*$ in $M$ ($\mathrm{da}_M(x)$) is the number of all accepting paths for $x$ in $M$, i.e. $\mathrm{da}_M(x)=\sum_{p\in Q_{\mathrm{I}}}\sum_{q\in Q_{\mathrm{F}}}\mathrm{da}_M(p,x,q)$. The *degree of ambiguity* of $M$ ($\mathrm{da}(M)$) is the supremum of the set $\{\mathrm{da}_M(x)\,|\,x\in\Sigma^*\}$. Clearly, for all $x\in\Sigma^*$, $\mathrm{da}_M(x)$ is at most $(\#Q)^{|x|+1}$. $M$ is called *infinitely ambiguous* (*finitely ambiguous, unambiguous*), if $\mathrm{da}(M)=\infty(<\infty, \leqslant 1)$.

A state of $M$ is called *useful* if it appears on some accepting path in $M$; otherwise, this state is called *useless*. Useless states are irrelevant to the degree of ambiguity in $M$. If all states of $M$ are useful, then $M$ is called *trim*.

A state $p\in Q$ is said to be *connected* with a state $q\in Q$ ($p\underset{M}{\leftrightarrow}q$) if some paths in $M$ lead from $p$ to $q$ and from $q$ to $p$. An equivalence class w.r.t. the relation "$\underset{M}{\leftrightarrow}$" is called *strong component* of $M$. A transition $(p,a,q)$ of $M$ is called a *bridge* if $p$ is not connected with $q$.

$M$ is said to be a *chain NFA* if, for some order $Q_1,\ldots,Q_k$ of the strong components of $M$, $(p_1,\ldots,p_k)$, $(q_1,\ldots,q_k)\in Q_1\times\cdots\times Q_k$ exist such that, in $M$, $p_1$ ($q_k$) is the only possible initial (final) state and every bridge is of the form $(q_i,a,p_{i+1})$, where $i\in[k-1]$ ($[m]$ denotes the set $\{1,\ldots,m\}$) and $a\in\Sigma$ (see Fig. 1). Let $M$ be a chain NFA such that $L(M)\neq\emptyset$. Then, $M$ is trim, and $Q_{\mathrm{I}}=\{p_1\}$, $Q_{\mathrm{F}}=\{q_k\}$.

Let $x=x_1\ldots x_m\in\Sigma^*(x_1,\ldots,x_m\in\Sigma)$. The *graph of accepting paths* for $x$ in $M$ ($G_M(x)$) is the directed graph $(V,E)$, where

- $V:=\{(q,j)\in Q\times\{0,\ldots,m\}\,|\,\exists q_{\mathrm{I}}\in Q_{\mathrm{I}}\,\exists q_{\mathrm{F}}\in Q_{\mathrm{F}}\colon (q_{\mathrm{I}},x_1\ldots x_j,q)\in\delta$
  $\&\ (q,x_{j+1}\ldots x_m,q_{\mathrm{F}})\in\delta\}$,
- $E:=\{((p,j-1),(q,j))\in V^2\,|\,j\in[m]\ \&\ (p,x_j,q)\in\delta\}$.

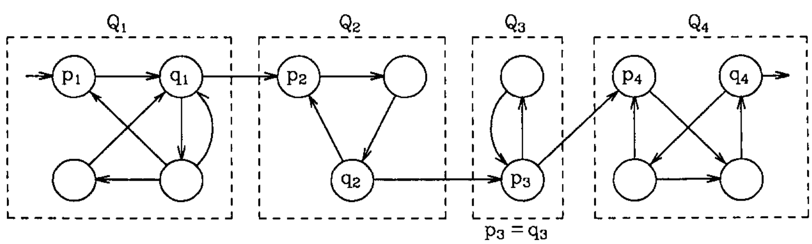



Fig. 1

Note: The number of all paths in $G_M(x)$ leading from $Q_I \times \{0\}$ to $Q_F \times \{m\}$ equals the degree of ambiguity of $x$ in $M$. Each vertex of $G_M(x)$ is situated on such a path.

Let $M = (Q, \Sigma, \delta, Q_I, Q_F)$ and $M' = (Q', \Sigma', \delta', Q'_I, Q'_F)$ be two NFAs. $M'$ is *included* in $M$ ($M' \subseteq M$) if inclusion holds in each component.

## 2. An upper bound for the finite degree of ambiguity

Let $M$ be a finitely ambiguous NFA with $n$ states. In this section we show the following:

(1) There are chain NFAs $M_1, \ldots, M_N \subseteq M$ such that $N \leqslant 5^{n/2}$ and $\mathrm{da}(M) \leqslant \sum_{i=1}^{N} \mathrm{da}(M_i) < \infty$.

(2) If $M$ is a chain NFA, then $\mathrm{da}(M) \leqslant n^n$.

From (1) and (2) follows Theorem 2.1.

**Theorem 2.1.** *Let $M$ be a finitely ambiguous* NFA *with $n$ states. Then, the degree of ambiguity of $M$ is at most $5^{n/2} \cdot n^n$.*

Using an estimate of Baron [2] the upper bound in Theorem 2.1 can be improved to $2^{1+k_2 \cdot n} \cdot n^n$, where $k_2 < 0.7956$ (note that $5^{n/2} = 2^{k_1 \cdot n}$, where $k_1 := (\log_2 5)/2 \approx 1.1610$). In Section 5 we state that each further improvement of this upper bound has to stop at $2^{1.0221 \cdot n}$ (see Theorem 5.1). By reduction, Theorem 2.1 can be generalized to NFAs with ε-moves [20, Theorem 6.1]. Moreover, it can be applied to finitely generated monoids of matrices with entries in $\mathbb{N}_0$ ([20, Theorems 7.1 and 7.3]; see [23, Appendix]).

Given an NFA $M$ and an integer $i$, it is decidable in polynomial space whether or not $\mathrm{da}(M)$ is greater than $i$ [3]. Thus, Theorem 2.1 implies that the degree of ambiguity of an NFA can be computed in polynomial space.

Our first lemma will show (1). In order to prove this lemma we need the following proposition.

**Proposition 2.2.** *Let $n = \sum_{i=1}^{k} n_i$ where $n_1, \ldots, n_k \in \mathbb{N}_0$. Then, $\prod_{i=1}^{k} (n_i^2 + 1) \leqslant 5^{n/2}$.*

**Proof.** It is easy to show by induction on $j$ that for each $j \in \mathbb{N}_0$ $(j^2+1)^2 \leqslant 5^j$. In turn, this implies that $\prod_{i=1}^{k} (n_i^2 + 1) \leqslant \prod_{i=1}^{k} 5^{n_i/2} = 5^{n/2}$. □

**Lemma 2.3.** *Let $M$ be an* NFA *with $n$ states and input alphabet $\Sigma$. There are chain* NFAs *$M_1, \ldots, M_N \subseteq M$ such that $N \leqslant 5^{n/2}$ and the following assertions are true:*

(i) $\forall x \in \Sigma^*$: $\mathrm{da}_M(x) = \sum_{i=1}^{N} \mathrm{da}_{M_i}(x)$.

(ii) $\mathrm{da}(M) < \infty \Rightarrow \mathrm{da}(M) \leqslant \sum_{i=1}^{N} \mathrm{da}(M_i) < \infty$.

(iii) $\mathrm{da}(M) = \infty \Rightarrow \exists i \in [N]: \mathrm{da}(M_i) = \infty$.

**Proof.** Let $M = (Q, \Sigma, \delta, Q_{\mathrm{I}}, Q_{\mathrm{F}})$. Let $Q_1, \ldots, Q_k$ be an order of the strong components of $M$ so that for all $i, j \in [k]$ the following holds:

$$\delta \cap Q_i \times \Sigma^* \times Q_j \neq \emptyset \Rightarrow i \leqslant j.$$

Let $K$ be a nonempty subset of $[k]$, let $1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant k$ so that $K = \{i_1, \ldots, i_l\}$, and let $\boldsymbol{p} = (p_{i_1}, \ldots, p_{i_l})$, $\boldsymbol{q} = (q_{i_1}, \ldots, q_{i_l}) \in Q^{(K)} := Q_{i_1} \times \cdots \times Q_{i_l}$. We construct the NFA $M^{(\boldsymbol{p}, \boldsymbol{q}, K)} = (\bigcup_{\lambda=1}^{l} Q_{i_\lambda}, \Sigma, \delta^{(\boldsymbol{p}, \boldsymbol{q}, K)}, Q_{\mathrm{I}}^{(\boldsymbol{p}, \boldsymbol{q})}, Q_{\mathrm{F}}^{(\boldsymbol{p}, \boldsymbol{q})})$:

$$Q_{\mathrm{I}}^{(\boldsymbol{p}, \boldsymbol{q})} := Q_{\mathrm{I}} \cap \{p_{i_1}\}, \qquad Q_{\mathrm{F}}^{(\boldsymbol{p}, \boldsymbol{q})} := Q_{\mathrm{F}} \cap \{q_{i_l}\},$$

$$\delta^{(\boldsymbol{p}, \boldsymbol{q}, K)} := \delta \cap \left( \bigcup_{\lambda=1}^{l} Q_{i_\lambda} \times \Sigma \times Q_{i_\lambda} \cup \bigcup_{\lambda=1}^{l-1} \{q_{i_\lambda}\} \times \Sigma \times \{p_{i_{\lambda+1}}\} \right).$$

$M^{(\boldsymbol{p}, \boldsymbol{q}, K)}$ is a chain NFA, which is included in $M$. For all $x \in \Sigma^*$ we observe that

$$\mathrm{da}_M(x) = \sum_{\emptyset \neq K \subseteq [k]} \sum_{\boldsymbol{p}, \boldsymbol{q} \in Q^{(K)}} \mathrm{da}_{M^{(\boldsymbol{p}, \boldsymbol{q}, K)}}(x).$$

From Proposition 2.2 it follows that

$$\sum_{\emptyset \neq K \subseteq [k]} \sum_{\boldsymbol{p}, \boldsymbol{q} \in Q^{(K)}} 1 = \sum_{(\sigma_1, \ldots, \sigma_k) \in \{0,1\}^k} \left( \prod_{i=1}^{k} (\# Q_i)^{2 \cdot \sigma_i} \right) - 1$$
$$= \prod_{i=1}^{k} ((\# Q_i)^2 + 1) - 1 \leqslant 5^{n/2} - 1 < 5^{n/2}.$$

This completes the proof of (i). The assertions (ii) and (iii) follow from (i). $\square$

Note that in Lemma 2.3 we showed, compared to [22], a better result with a shorter proof. The two next lemmas will demonstrate (2).

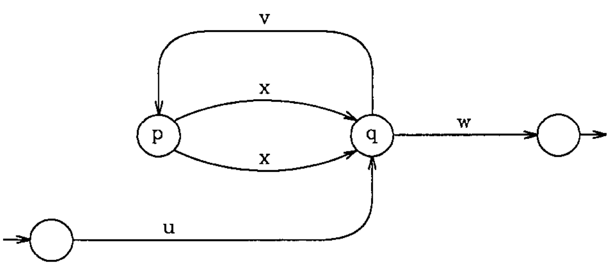



Fig. 2

**Lemma 2.4.** *Let $M=(Q, \Sigma, \delta, Q_{\mathrm{I}}, Q_{\mathrm{F}})$ be a finitely ambiguous* NFA. *Let $p, q \in Q$ be useful states so that $p$ is connected with $q$. Then, for all $x \in \Sigma^*$, $\mathrm{da}_M(p, x, q)$ is at most* 1.

**Proof.** Assume that for some $x \in \Sigma^*$ $\mathrm{da}_M(p, x, q) \geqslant 2$. Select $v \in \Sigma^*$ so that $(q, v, p) \in \delta$. Then, $u, w \in \Sigma^*$ exist such that for all $i \in \mathbb{N}_0$ $\mathrm{da}_M(u(vx)^i w) \geqslant \mathrm{da}_M(q, (vx)^i, q) \geqslant 2^i$ (see Fig. 2). Thus, $\mathrm{da}(M)=\infty$. (Contradiction!) □

**Lemma 2.5.** *Let $M$ be a finitely ambiguous chain* NFA *with $n$ states. Then, the degree of ambiguity of $M$ is at most $n^n$.*

**Proof.** Let $M=(Q, \Sigma, \delta, Q_{\mathrm{I}}, Q_{\mathrm{F}})$. Let $Q_1, \ldots, Q_k \subseteq Q$ and $(p_1, \ldots, p_k)$, $(q_1, \ldots, q_k) \in Q_1 \times \cdots \times Q_k$ be given in correspondence with the definition of a chain NFA. Let w.l.o.g. $\mathrm{da}(M)>0$. Then, $M$ is trim, and $Q_{\mathrm{I}}=\{p_1\}$, $Q_{\mathrm{F}}=\{q_k\}$. We will show by induction on $k$ that

$$(*) \qquad \mathrm{da}(M) \leqslant 2^{n \cdot \lceil \log_2 k \rceil - k + 1}.$$

First of all, we show that the lemma follows from $(*)$. Define $t_0 := \lceil \log_2 n \rceil$.

*Case 1*: $k \in \{1, \ldots, 2^{t_0-1}\}$. Then, $n \cdot \lceil \log_2 k \rceil - k + 1 \leqslant n \cdot (t_0 - 1) < n \cdot \log_2 n$.

*Case 2*: $k \in \{2^{t_0-1}+1, \ldots, n\}$. Then, $n \cdot \lceil \log_2 k \rceil - k + 1 \leqslant n \cdot t_0 - 2^{t_0-1}$. Consider the function $f$ which maps $v$ to $v \cdot \log_2 v - (v \cdot t_0 - 2^{t_0-1})$. $f$ is continuous and differentiable on the open interval $(0, \infty)$. We know that $f'(v) = \log_2 v + 1/(\log_e 2) - t_0$. $f'$ is a monotonously ascending function. Moreover, $f'(2^{t_0-1})>0$ and $n > 2^{t_0-1}$. From this follows with standard arguments that $f(n) \geqslant f(2^{t_0-1}) = 0$, i.e. $n \cdot t_0 - 2^{t_0-1} \leqslant n \cdot \log_2 n$.

**Proof of $(*)$.** *Base of induction*: $k=1$. $p_1$ is connected with $q_1$. Thus, according to Lemma 2.4, $M$ is unambiguous, i.e. $\mathrm{da}(M)=1$.

*Induction step*: Let $k \geqslant 2$. Define $l := \lceil k/2 \rceil$. We (uniquely) divide $M$ into the NFAs $M_1 = (\bigcup_{i=1}^{l} Q_i, \Sigma, \delta_1, \{p_1\}, \{q_l\})$ and $M_2 = (\bigcup_{i=l+1}^{k} Q_i, \Sigma, \delta_2, \{p_{l+1}\}, \{q_k\})$ so that $\delta = \delta_1 \cup \delta_2 \cup (\delta \cap \{q_l\} \times \Sigma \times \{p_{l+1}\})$. $M_1$ and $M_2$ are finitely ambiguous chain NFAs with $n_1 := \sum_{i=1}^{l} \#Q_i$ and $n_2 := \sum_{i=l+1}^{k} \#Q_i$ states, respectively, such that $\mathrm{da}(M_1)>0$ and $\mathrm{da}(M_2)>0$.

Let $x = x_1 \ldots x_m \in L(M)$ $(x_1, \ldots, x_m \in \Sigma)$. Consider in the graph $G_M(x)=(V, E)$ the set $D$ of all edges "leading from $Q_l$ to $Q_{l+1}$": $D := \{((q_l, j-1), (p_{l+1}, j)) \in E \mid j \in [m]\}$. Let $J \subseteq [m]$ so that $D = \{((q_l, j-1), (p_{l+1}, j)) \mid j \in J\}$. We observe (see Fig. 3) that

$$\mathrm{da}_M(x) = \sum_{j \in J} \mathrm{da}_{M_1}(x_1 \ldots x_{j-1}) \cdot \mathrm{da}_{M_2}(x_{j+1} \ldots x_m).$$

From this follows with the induction hypothesis that

$$\mathrm{da}_M(x) \leqslant \sum_{j \in J} 2^{n_1 \cdot \lceil \log_2 \lceil k/2 \rceil \rceil - \lceil k/2 \rceil + 1} \cdot 2^{n_2 \cdot \lceil \log_2 \lfloor k/2 \rfloor \rceil - \lfloor k/2 \rfloor + 1}$$

$$\leqslant \# J \cdot 2^{n \cdot \lceil \log_2 (k/2) \rceil - k + 2} = \# J \cdot 2^{n \cdot \lceil \log_2 k \rceil - n - k + 2}.$$

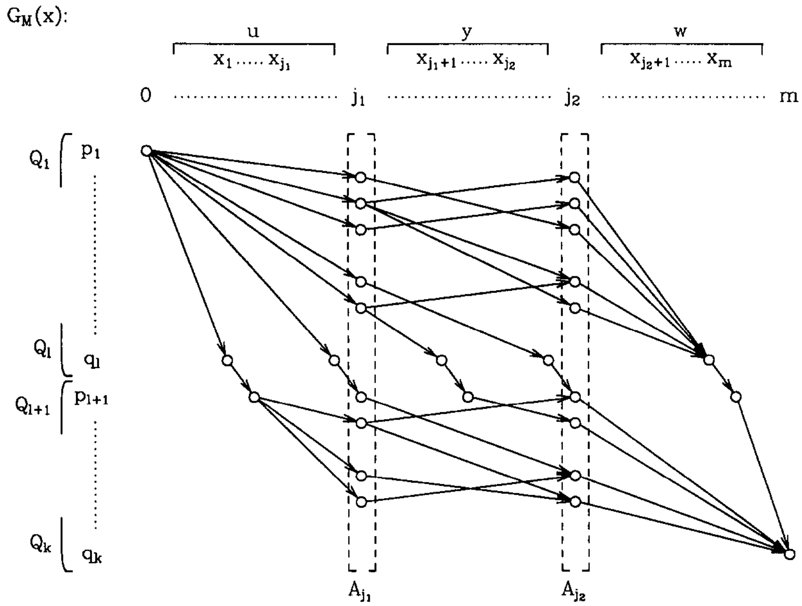



Fig. 3

Note that $\lceil \log_2 \lceil k/2 \rceil \rceil = \lceil \log_2(k/2) \rceil$. Therefore, in order to prove (*), it is sufficient to show that $\# D = \# J \leqslant 2^{n-1}$.

Assume that $\# J > 2^{n-1}$. Let $j \in J$. Define $A_j := \{q \in Q \mid (q, j) \in V\}$. Clearly, $p_{l+1} \in A_j \subseteq Q$. Since $\# J > 2^{n-1}$, $j_1, j_2 \in J$ exist such that $j_1 < j_2$ and $A_{j_1} = A_{j_2}$. Let us define $A := A_{j_1} = A_{j_2}$, $u := x_1 \ldots x_{j_1}$, $y_1 := x_{j_1+1} \ldots x_{j_2-1}$, $a_1 := x_{j_2}$, $y := y_1 a_1$, and $w := x_{j_2+1} \ldots x_m$. Then, we observe (see Fig. 3)

$$(q_l, a_1, p_{l+1}) \in \delta \;\; \& \;\; \exists p \in A: (p, y_1, q_l) \in \delta,$$

$$\forall q \in A: (p_1, u, q) \in \delta \;\; \& \;\; \exists p \in A: (p, y, q) \in \delta,$$

$$\forall p \in A: (p, w, q_k) \in \delta \;\; \& \;\; \exists q \in A: (p, y, q) \in \delta.$$

Let $t \in \mathbb{N}$. From the above expressions it follows that for all $\tau \in [t]$

$$(p_1, u y^{\tau-1} y_1, q_l) \in \delta, \qquad (q_l, a_1, p_{l+1}) \in \delta, \qquad (p_{l+1}, y^{t-\tau} w, q_k) \in \delta.$$

This implies that $\mathrm{da}_M(u y^t w) = \mathrm{da}_M(p_1, u y^t w, q_k) \geqslant t$ (see Fig. 3). Thus, $\mathrm{da}(M) = \infty$. (Contradiction!)

We remark that using an estimate of Baron [2] the upper bound in (*) can be slightly improved to $2^{n \cdot \lceil \log_2 k \rceil - 2^{\lceil \log_2 k \rceil} + 1}$. $\square$

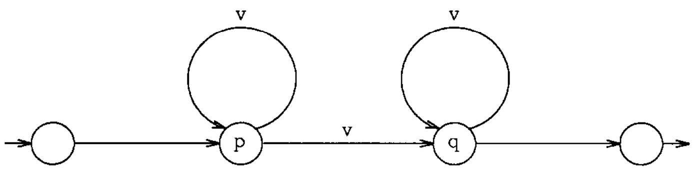



Fig. 4

## 3. A criterion for the infinite degree of ambiguity

Let $M=(Q,\Sigma,\delta,Q_{\mathrm{I}},Q_{\mathrm{F}})$ be an NFA. We introduce the following criterion (IDA) which characterizes the infinite degree of ambiguity of $M$:

(IDA): There are distinct useful states $p,q\in Q$ such that for some word $v\in\Sigma^*$ $(p,v,p),(p,v,q),(q,v,q)\in\delta$ (see Fig. 4).

Let $M$ comply with (IDA), let $p,q\in Q$ and $v\in\Sigma^*$ be selected according to that criterion. Then, $u,w\in\Sigma^*$ exist such that for all $i\in\mathbb{N}$ $\mathrm{da}_M(uv^iw)\geqslant\mathrm{da}_M(p,v^i,q)\geqslant i$. Thus, $\mathrm{da}(M)=\infty$. On the other hand, assume that $M$ is infinitely ambiguous. Then, according to Lemma 2.3, there is a chain NFA $M'\subseteq M$ which is infinitely ambiguous, too. In Lemma 3.3 we will show the following:

(3) If $M''$ is an infinitely ambiguous chain NFA, then it complies with (IDA).

According to (3), $M'$ complies with (IDA) and, hence, $M$ complies with (IDA), too. Therefore, we have shown that (3) implies Theorem 3.1.

**Theorem 3.1.** *Let $M$ be an* NFA. *$M$ is infinitely ambiguous if and only if it complies with* (IDA).

Let $M=(Q,\Sigma,\delta,Q_{\mathrm{I}},Q_{\mathrm{F}})$ be an NFA with $n$ states. We sketch an algorithm which decides in polynomial time whether or not $M$ complies with (IDA):

- Remove all useless states from $M$. Let w.l.o.g. $M$ be trim.
- For all $p,q\in Q$ with $p\neq q$ do the following:
  - Construct an NFA $M'$ such that $L(M')=L_1\cap L_2\cap L_3$, where

    $$L_1:=\{v\in\Sigma^*\,|\,(p,v,p)\in\delta\},\qquad L_2:=\{v\in\Sigma^*\,|\,(p,v,q)\in\delta\},$$
    $$L_3:=\{v\in\Sigma^*\,|\,(q,v,q)\in\delta\}.$$

  - Decide whether or not $L(M')$ is empty.

In fact, the above algorithm requires time $O(n^8\cdot\#\Sigma)$. In Lemma 3.4 we will show the following:

(4) It is decidable in time $O(n^6\cdot\#\Sigma)$ whether or not $M$ complies with (IDA).

Theorem 3.1 and assertion (4) imply Theorem 3.2.

**Theorem 3.2.** *Let $M$ be an* NFA *with $n$ states and input alphabet $\Sigma$. It is decidable in time $O(n^6 \cdot \#\Sigma)$ whether or not $M$ is infinitely ambiguous.*

By reduction, Theorem 3.2 can be generalized to NFAs with $\varepsilon$-moves [20, Theorem 6.2]. Moreover, it implies a polynomial-time algorithm deciding whether or not a finitely generated monoid of matrices with entries in $\mathbb{N}_0$ is finite ([20, Theorem 7.2]; see [23, Appendix]).

The following lemma shows (3) and, thus, completes the proof of Theorem 3.1.

**Lemma 3.3.** *Let $M$ be an infinitely ambiguous chain* NFA. *Then, $M$ complies with* (IDA).

**Proof.** Let $M=(Q,\Sigma,\delta,Q_I,Q_F)$. Let $Q_1,\ldots,Q_k\subseteq Q$ and $(p_1,\ldots,p_k),(q_1,\ldots,q_k)\in Q_1\times\cdots\times Q_k$ be given in correspondence with the definition of a chain NFA. Since $\mathrm{da}(M)=\infty$, we know that $M$ is trim and $Q_I=\{p_1\}, Q_F=\{q_k\}$.

*Case 1*: $\exists(p',y,q')\in Q\times\Sigma^*\times Q$: $p' \underset{M}{\leftrightarrow} q'$ & $\mathrm{da}_M(p',y,q')\geqslant 2$.

Then, distinct states $p,q\in Q$ and words $y_1,y_2,y_3\in\Sigma^*$ exist so that $y=y_1y_2$, $(p',y_1,p),(p',y_1,q)\in\delta$, $(p,y_2,q'),(q,y_2,q')\in\delta$, and $(q',y_3,p')\in\delta$ (see Fig. 5). Define $v:=y_2y_3y_1$, then $\{p,q\}\times\{v\}\times\{p,q\}\subseteq\delta$. Hence, since $p$ and $q$ are useful, $M$ complies with (IDA).

*Case 2*: $\forall(p',y,q')\in\bigcup_{i=1}^k Q_i\times\Sigma^*\times Q_i$: $\mathrm{da}_M(p',y,q')\leqslant 1$.

Let $x=x_1\ldots x_m\in L(M)$ $(x_1,\ldots,x_m\in\Sigma)$. Consider $G_M(x)=(V,E)$. Let $l\in[k-1]$ (note that $k\geqslant 2$). Define $D_l(x):=\{((q_l,j-1),(p_{l+1},j))\in E \mid j\in[m]\}$. $D_l(x)$ is the set of all edges in $G_M(x)$ "leading from $Q_l$ to $Q_{l+1}$". According to Case 2 we know that

$$(\#)\qquad \mathrm{da}_M(x)\leqslant \#(D_1(x)\times\cdots\times D_{k-1}(x))=\prod_{l=1}^{k-1}\#D_l(x).$$

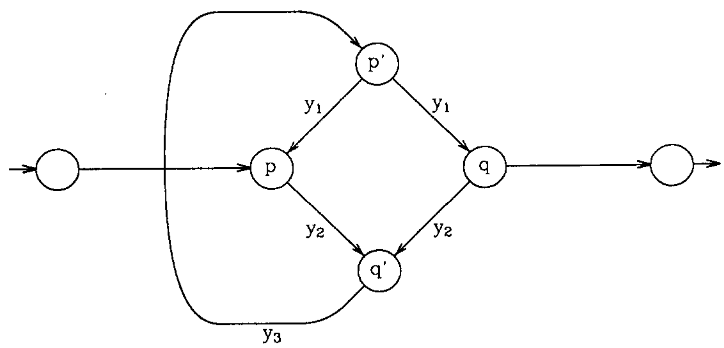



Fig. 5

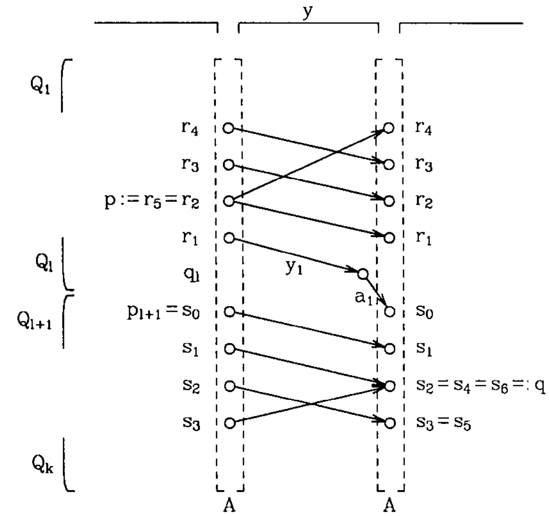



Fig. 6

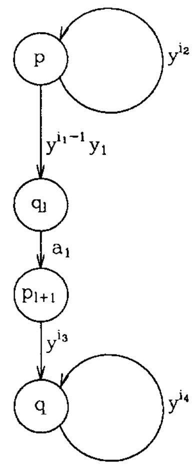



Fig. 7

Set $n := \#Q$. Since $\mathrm{da}(M)=\infty$, we may w.l.o.g. assume that $\mathrm{da}_M(x)>2^{(n-1)\cdot(k-1)}$. Because of $(\#)$ we are able to choose $l\in[k-1]$ so that $\#D_l(x)>2^{n-1}$.

The following construction is performed just like in the proof of Lemma 2.5. Let $J\subseteq[m]$ so that $D_l(x)=\{((q_l,j-1),(p_{l+1},j))\,|\,j\in J\}$. Let $j\in J$. Define $A_j:=\{r\in Q\,|\,(r,j)\in V\}$. Clearly, $p_{l+1}\in A_j\subseteq Q$. Since $\#D_l(x)>2^{n-1}$, $j_1,j_2\in J$ exist such that $j_1<j_2$ and $A_{j_1}=A_{j_2}$. Let us define $A:=A_{j_1}=A_{j_2}$, $y_1:=x_{j_1+1}\ldots x_{j_2-1}$, $a_1:=x_{j_2}$, and $y:=y_1a_1$. Then, we observe that

$$(q_l,a_1,p_{l+1})\in\delta \;\;\&\;\; \exists r\in A\colon (r,y_1,q_l)\in\delta,$$

$$\forall s\in A\;\exists r\in A\colon (r,y,s)\in\delta \;\;\&\;\; \forall r\in A\;\exists s\in A\colon (r,y,s)\in\delta.$$

We construct states $r_i\in A\,(i\geqslant 1)$ as follows. Choose $r_1\in A$ so that $(r_1,y_1,q_l)\in\delta$. Choose $r_i\in A$ so that $(r_i,y,r_{i-1})\in\delta$ $(i=2,3,\ldots)$. There are $i_1$, $i_2\in\mathbb{N}$ such that $r_{i_1}=r_{i_1+i_2}=:p$ (see Fig. 6). We construct states $s_i\in A$ $(i\geqslant 0)$ as follows. Define $s_0:=p_{l+1}$. Choose $s_i\in A$ so that $(s_{i-1},y,s_i)\in\delta$ and for all $j\in[i-1]$ $s_{i-1}=s_{j-1}$ implies $s_i=s_j$ $(i=1,2,\ldots)$. There are $i_3\in\mathbb{N}_0$ and $i_4\in\mathbb{N}$ such that $s_{i_3}=s_{i_3+i_4}=:q$ and $i_1+i_3=0 \bmod i_2\cdot i_4$ (see Fig. 6). In conclusion, we have (see Fig. 7)

$$(p,y^{i_2},p)\in\delta,\quad (p,y^{i_1-1}y_1,q_l),(q_l,a_1,p_{l+1}),(p_{l+1},y^{i_3},q)\in\delta,\quad (q,y^{i_4},q)\in\delta.$$

Since $q_l$ is not connected with $p_{l+1}$, $p$ and $q$ must be distinct. Let $j_1\in\mathbb{N}$ so that $i_1+i_3=j_1\cdot i_2\cdot i_4$, and define $v:=y^{j_1\cdot i_2\cdot i_4}$. Then, $(p,v,p),(p,v,q),(q,v,q)\in\delta$. Hence, since $p$ and $q$ are useful, $M$ complies with (IDA). $\square$

We remark that, after some slight modifications, the above proof also works for nonchain NFAs. Thus, Theorem 3.1 can be proved without using Lemma 2.3. Indeed, the counting mechanism of this lemma, although quite elementary, is much too fine for our purpose here.

In order to show (4) we need some preliminaries. In a finite, directed graph $G=(V,E)$ we use the following notations: Let $p,q\in V$. We write $p \underset{G}{\rightarrow} q$, if some path in $G$ leads from $p$ to $q$. We write $p \underset{G}{\leftrightarrow} q$, if some paths in $G$ lead from $p$ to $q$ and from $q$ to $p$. $p \underset{G}{\leftrightarrow} q$ means that $p$ is strongly connected with $q$. An equivalence class w.r.t. the relation "$\underset{G}{\leftrightarrow}$" is called a *strong component* of $G$.

Let $M=(Q,\Sigma,\delta,Q_{\mathrm{I}},Q_{\mathrm{F}})$ be an NFA. We define the directed graphs $G_3=(Q^3,E_3)$ and $G_4=(Q^3,E_4)$:

$$E_3:=\{((p_1,p_2,p_3),(q_1,q_2,q_3))\in Q^3\times Q^3 \mid \exists a\in\Sigma\ \forall i\in\{1,2,3\}: (p_i,a,q_i)\in\delta\},$$

$$E_4:=E_3\cup E'_3,\text{ where } E'_3:=\{((p,q,q),(p,p,q)) \mid p,q\in Q, p\neq q\}.$$

Let $M$ be trim. $G_3$ allows to rewrite (IDA) as follows:

(∗) There are distinct states $p,q\in Q$ such that $(p,p,q) \underset{G_3}{\rightarrow} (p,q,q)$.

It is easy to verify that (∗) is equivalent to (IDA)′.

(IDA)′: There is a strong component $U$ of $G_4$ so that $U^2\cap E'_3\neq\emptyset$.

The following lemma shows (4) and, thus, completes the proof of Theorem 3.2.

**Lemma 3.4.** *Let* $M=(Q,\Sigma,\delta,Q_{\mathrm{I}},Q_{\mathrm{F}})$ *be an* NFA *with* $n$ *states. It is decidable in worst-case time* $\mathrm{O}(n^6\cdot\#\Sigma)$ *(on a RAM without multiplications and divisions using the uniform cost criterion) whether or not* $M$ *complies with (IDA).*

**Proof.** For background information on RAMs we refer to [15] and [1]. We present an informal algorithm deciding whether or not $M$ complies with (IDA). Note that this algorithm uses well-known graph algorithms (see [1]) as subroutines:

Step 1: Remove all useless states from $M$. Let w.l.o.g. $M$ be trim.
Step 2: Construct $E'_3$ and $G_4$.
Step 3: Compute the strong components of $G_4$.
Step 4: Decide whether or not $M$ complies with (IDA)′, i.e. check whether or not there is a strong component $U$ of $G_4$ so that $U^2\cap E'_3\neq\emptyset$.

It can be easily seen that the above algorithm has worst-case time complexity $\mathrm{O}(n^6\cdot\#\Sigma)$. □

We add that a variant of the algorithm presented in the above proof can be implemented on a nondeterministic logarithmically space-bounded Turing machine (see, e.g. [6]). Thus, the problem "decide whether or not an NFA is infinitely ambiguous" belongs to NSPACE($\log_2 n$) and, hence, also to NC.

## 4. The degree of growth of the ambiguity

Let $M=(Q,\Sigma,\delta,Q_I,Q_F)$ be an NFA. The *degree of growth of the ambiguity* of $M$ ($\deg(M)$) is defined as the minimal degree of a polynomial $h\in\mathbb{N}_0[X]$ such that for all $x\in\Sigma^*$ $\mathrm{da}_M(x)$ is at most $h(|x|)$ if such a polynomial exists, or is infinite otherwise. If $\deg(M)$ is finite (infinite), then $M$ is called *polynomially ambiguous* (*exponentially ambiguous*). By definition, $\deg(M)=0$ iff $\mathrm{da}(M)<\infty$, and $\deg(M)\geqslant 1$ iff $\mathrm{da}(M)=\infty$. Thus, the degree of growth of the ambiguity allows to distinguish infinitely ambiguous NFAs.

The following criterion (EDA) characterizes the property "$M$ is exponentially ambiguous":

(EDA): There is a useful state $q\in Q$ such that, for some word $v\in\Sigma^*$, $\mathrm{da}_M(q,v,q)\geqslant 2$ (see Fig. 8).

Let $d\in\mathbb{N}$. The following criterion ($\mathrm{IDA}_d$) characterizes the property "$\deg(M)\geqslant d$":

($\mathrm{IDA}_d$): There are useful states $r_1,s_1,\ldots,r_d,s_d\in Q$ and words $v_1,u_2,v_2,\ldots,u_d,v_d\in\Sigma^*$ such that for all $\lambda=1,\ldots,d$ $r_\lambda$ and $s_\lambda$ are distinct and $(r_\lambda,v_\lambda,r_\lambda),(r_\lambda,v_\lambda,s_\lambda),(s_\lambda,v_\lambda,s_\lambda)\in\delta$ and for all $\lambda=2,\ldots,d$ $(s_{\lambda-1},u_\lambda,r_\lambda)\in\delta$ (see Fig. 9).

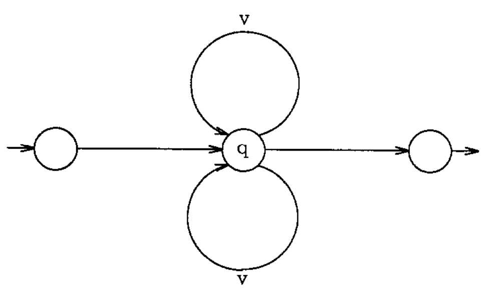



Fig. 8

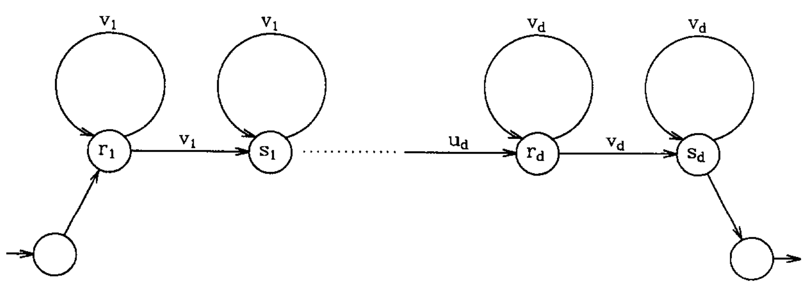



Fig. 9

*Note*: ($IDA_1$) equals the criterion (IDA) introduced in Section 3. Let $d \in \mathbb{N}$. If $M$ complies with ($IDA_d$), then it also complies with ($IDA_1$), ..., ($IDA_{d-1}$). If, moreover, for some $\lambda \in [d]$, $r_\lambda$ is connected with $s_\lambda$, where $r_\lambda, s_\lambda \in Q$ are selected according to ($IDA_d$), then $M$ complies with (EDA).

In Section 3 we have shown that $\deg(M) \geqslant 1$ if and only if $M$ complies with ($IDA_1$). In this section we show the following theorems.

**Theorem 4.1.** (Reutenauer [16], Ibarra and Ravikumar [7]). *Let $M$ be an* NFA *with $n$ states. The assertions* (i)–(iii) *are equivalent.*

(i) *$M$ is exponentially ambiguous.*
(ii) $\deg(M) \geqslant n$.
(iii) *$M$ complies with* (EDA).

**Theorem 4.2.** *Let $M$ be an* NFA, *and let $d \in \mathbb{N}$. The assertions* (i) *and* (ii) *are equivalent.*

(i) $\deg(M) \geqslant d$.
(ii) *$M$ complies with* ($IDA_d$).

*Note*: Our proof of Theorem 4.1 is mostly different from those in [16] and [7]. Theorem 4.2 is a generalization of Theorem 3.1.

Let $M = (Q, \Sigma, \delta, Q_I, Q_F)$ be an NFA with $n$ states. We want to decide whether or not $M$ complies with (EDA). For this we define the directed graph $G_2 = (Q^2, E_2)$:

$$E_2 := \{((p_1, p_2), (q_1, q_2)) \in Q^2 \times Q^2 \mid \exists a \in \Sigma\ \forall i \in \{1, 2\}: (p_i, a, q_i) \in \delta\}.$$

Let $M$ be trim. Then, (EDA) is equivalent to (EDA)′:

(EDA)′: There is a strong component of $G_2$ which contains vertices $(p_1, p_2)$ and $(q_1, q_2)$ so that $p_1 = p_2$ and $q_1 \neq q_2$.

Thus, using a straightforward algorithm (cf. Lemma 3.4), the following result can be easily established:

(5) It is decidable in time $O(n^4 \cdot \#\Sigma)$ whether or not $M$ complies with (EDA).

Theorem 4.1 and assertion (5) imply the following theorem.

**Theorem 4.3.** *Let $M$ be an* NFA *with $n$ states and input alphabet $\Sigma$. It is decidable in time $O(n^4 \cdot \#\Sigma)$ whether or not $M$ is exponentially ambiguous.*

Let $M = (Q, \Sigma, \delta, Q_I, Q_F)$ be an NFA with $n$ states. We want to compute $\deg(M)$. Theorem 4.2 implies that

(6) $\deg(M) = \sup(\{0\} \cup \{d \in \mathbb{N} \mid M \text{ complies with } (IDA_d)\})$.

By topological sort we can find an order $Q_1, \ldots, Q_k$ of the strong components of $M$ so that for all $i, j \in [k]$ the following holds:

$$\delta \cap Q_i \times \Sigma^* \times Q_j \neq \emptyset \Rightarrow i \leqslant j.$$

Using the notation introduced in Section 3, we define the directed graph $G_5 = (V_5, E_5)$ and the set $R \subseteq E_5$:

$$V_5 := \{Q_1, \ldots, Q_k\},$$

$$R := \{(Q_i, Q_j) \in V_5 \times V_5 \mid \text{there is a strong component } U \text{ of } G_4 \text{ so that}$$
$$U^2 \cap E'_3 \neq \emptyset \text{ and } U \cap \{(p, p, q) \in Q^3 \mid p \neq q, p \in Q_i, q \in Q_j\} \neq \emptyset\},$$

$$E_5 := R \cup \{(Q_i, Q_j) \in V_5 \times V_5 \mid \delta \cap Q_i \times \Sigma^* \times Q_j \neq \emptyset\}.$$

Let $i, j \in [k]$. It is easy to verify that $(Q_i, Q_j) \in R$ if and only if there are distinct states $p \in Q_i$ and $q \in Q_j$ such that for some word $v \in \Sigma^*$ $(p, v, p), (p, v, q), (q, v, q) \in \delta$. In particular, if $(Q_i, Q_j) \in E_5$, then $i \leqslant j$.

Let $M$ be trim. Then, we know for all $d \in \mathbb{N}$ that $M$ complies with $(\mathrm{IDA}_d)$ if and only if there is a path in $G_5$ which contains $d$ edges of $R$. Hence, we conclude from (6) that either $R \cap \{(Q_i, Q_i) \mid i \in [k]\} \neq \emptyset$, which implies that $\deg(M) = \infty$, or $\deg(M)$ is the maximal number of edges of $R$ on any path in $G_5$. Thus, using a straightforward algorithm (cf. Lemma 3.4), the following result can be easily established.

**Theorem 4.4.** *Let $M$ be an* NFA *with $n$ states and input alphabet $\Sigma$. Then,* $\deg(M)$ *is computable in time* $\mathrm{O}(n^6 \cdot \#\Sigma)$.

Note that Theorem 4.4 generalizes Theorem 3.2.

In the rest of this section we consider an NFA $M = (Q, \Sigma, \delta, Q_\mathrm{I}, Q_\mathrm{F})$. In order to prove Theorem 4.1 we need the following lemma.

**Lemma 4.5.** *Let $M$ be a chain* NFA *with $k$ strong components which does not comply with* (EDA). *Then,* $\deg(M) \leqslant k - 1$.

**Proof.** Let $Q_1, \ldots, Q_k \subseteq Q$ and $(p_1, \ldots, p_k), (q_1, \ldots, q_k) \in Q_1 \times \cdots \times Q_k$ be given in correspondence with the definition of a chain NFA. Let w.l.o.g. $L(M) \neq \emptyset$. Then, $M$ is trim, and $Q_\mathrm{I} = \{p_1\}$, $Q_\mathrm{F} = \{q_k\}$. We show by induction on $k$ that

$(*)$ $\quad \forall x \in \Sigma^*\ \mathrm{da}_M(x) \leqslant |x|^{k-1}$.

From $(*)$ it follows that $\deg(M) \leqslant k - 1$.

**Proof of $(*)$.** *Base of induction*: $k = 1$. Select $v \in \Sigma^*$ so that $(q_1, v, p_1) \in \delta$. Assume that, for some $x \in \Sigma^*$, $\mathrm{da}_M(x) = \mathrm{da}_M(p_1, x, q_1) \geqslant 2$. Then, $\mathrm{da}_M(q_1, vx, q_1) \geqslant 2$. Hence, since $q_1$ is useful, $M$ complies with (EDA). (Contradiction!)

*Induction step*: Let $k \geq 2$. We (uniquely) divide $M$ into the NFAs $M_1 = (Q_1, \Sigma, \delta_1, \{p_1\}, \{q_1\})$ and $M_2 = (\bigcup_{i=2}^{k} Q_i, \Sigma, \delta_2, \{p_2\}, \{q_k\})$ so that $\delta = \delta_1 \cup \delta_2 \cup (\delta \cap \{q_1\} \times \Sigma \times \{p_2\})$. $M_1$ and $M_2$ are chain NFAs which do not comply with (EDA). Moreover, $L(M_i) \neq \emptyset$ $(i = 1, 2)$. Let $x = x_1 \ldots x_m \in L(M)$ $(x_1, \ldots, x_m \in \Sigma)$. We observe that

$$\mathrm{da}_M(x) \leq \sum_{j=1}^{m} \mathrm{da}_{M_1}(x_1 \ldots x_{j-1}) \cdot \mathrm{da}_{M_2}(x_{j+1} \ldots x_m).$$

From this follows with the induction hypothesis that

$$\mathrm{da}_M(x) \leq \sum_{j=1}^{m} |x_{j+1} \ldots x_m|^{k-2} \leq m \cdot |x|^{k-2} = |x|^{k-1}.$$

Hence, we know for all $x \in \Sigma^*$ that $\mathrm{da}_M(x) \leq |x|^{k-1}$. $\square$

**Proof of Theorem 4.1.** Let $M$ be an NFA with $n$ states.

(i) $\Rightarrow$ (ii): trivial.

(ii) $\Rightarrow$ (iii): Let $\deg(M) \geq n$. According to Lemma 2.3, there is a chain NFA $M' \subseteq M$ (having $k'$ strong components) such that $\deg(M') \geq n$. Since $\deg(M') \geq n \geq k'$, Lemma 4.5 implies that $M'$ complies with (EDA). Therefore, $M$ complies with (EDA), too.

(iii) $\Rightarrow$ (i): Let $M$ comply with (EDA). Let $q \in Q$ and $v \in \Sigma^*$ be selected according to that criterion. Then, $u, w \in \Sigma^*$ exist so that for all $i \in \mathbb{N}_0$ $\mathrm{da}_M(uv^iw) \geq \mathrm{da}_M(q, v^i, q) \geq 2^i$. Assume that $\deg(M)$ is finite. Then, there is a polynomial $h \in \mathbb{N}_0[X]$ such that for all $i \in \mathbb{N}_0$ the following holds: $2^i \leq \mathrm{da}_M(uv^iw) \leq h(|uw| + |v| \cdot i)$. (Contradiction!) Hence, $M$ is exponentially ambiguous. $\square$

In order to prove Theorem 4.2 we need the following lemma.

**Lemma 4.6.** *Let $M$ be a chain* NFA *which does not comply with* (EDA). *Let $d \in \mathbb{N}$ so that* $\deg(M) \geq d$. *Then, $M$ complies with* ($\mathrm{IDA}_d$).

**Proof.** Let $Q_1, \ldots, Q_k \subseteq Q$ and $(p_1, \ldots, p_k), (q_1, \ldots, q_k) \in Q_1 \times \cdots \times Q_k$ be given in correspondence with the definition of a chain NFA. Since $\deg(M) \geq 1$, we know that $M$ is trim and $Q_\mathrm{I} = \{p_1\}$, $Q_\mathrm{F} = \{q_k\}$. We prove the lemma by induction on $k$.

*Base of induction*: $k = 1$. From Lemma 4.5 (for $k = 1$) follows that $\deg(M) = 0$.

*Induction step*: Let $k \geq 2$. For each $\lambda \in [k]$ we construct the NFA $M_\lambda = (\bigcup_{i=1}^{\lambda} Q_i, \Sigma, \delta_\lambda, \{p_1\}, \{q_\lambda\})$:

$$\delta_\lambda := \delta \cap \left( \bigcup_{i=1}^{\lambda} Q_i \times \Sigma \times Q_i \cup \bigcup_{i=1}^{\lambda-1} \{q_i\} \times \Sigma \times \{p_{i+1}\} \right).$$

From Lemma 4.5 (for $k = 1$) follows that $\deg(M_1) = 0$. Since $\deg(M_k) = \deg(M) \geq 1$, there is an $l \in [k-1]$ such that $\deg(M_l) = 0$ and $\deg(M_{l+1}) \geq 1$. According to

Theorem 3.1, $M_{l+1}$ complies with ($\text{IDA}_1$). Thus, if $d=1$, the lemma is already proved (since $M$ also complies with ($\text{IDA}_1$)). Therefore, let $d \geqslant 2$.

Let $M' = (\bigcup_{i=l+1}^{k} Q_i, \Sigma, \delta', \{p_{l+1}\}, \{q_k\})$ be the uniquely determined NFA so that $\delta = \delta_1 \cup \delta' \cup (\delta \cap \{q_l\} \times \Sigma \times \{p_{l+1}\})$. $M'$ is a chain NFA which does not comply with (EDA). Assume that $\deg(M') \leqslant d-2$. Then, there is a polynomial $h' \in \mathbb{N}_0[X]$ of degree $d-2$ such that for all $x \in \Sigma^*$ $\text{da}_{M'}(x)$ is at most $h'(|x|)$. Let $x = x_1 \ldots x_m \in L(M)$ ($x_1, \ldots, x_m \in \Sigma$). We observe that

$$\text{da}_M(x) \leqslant \sum_{j=1}^{m} \text{da}_{M_l}(x_1 \ldots x_{j-1}) \cdot \text{da}_{M'}(x_{j+1} \ldots x_m)$$

$$\leqslant \sum_{j=1}^{m} \text{da}(M_l) \cdot h'(|x_{j+1} \ldots x_m|) \leqslant \text{da}(M_l) \cdot |x| \cdot h'(|x|).$$

This implies that $\deg(M) \leqslant d-1$. (Contradiction!) Hence, $\deg(M') \geqslant d-1$. From the induction hypothesis it follows that $M'$ complies with ($\text{IDA}_{d-1}$). Hence, since $M_{l+1}$ complies with ($\text{IDA}_1$), $M$ complies with ($\text{IDA}_d$). $\square$

**Proof of Theorem 4.2.** Let $M$ be an NFA, and let $d \in \mathbb{N}$.

(i) $\Rightarrow$ (ii): Let $\deg(M) \geqslant d$.

*Case 1*: $M$ complies with (EDA).

Let $q \in Q$ and $v \in \Sigma^*$ be selected according to (EDA). Then, distinct states $r, s \in Q$ and words $y_1, y_2 \in \Sigma^*$ exist so that $v = y_1 y_2$, $(q, y_1, r), (q, y_1, s) \in \delta$, and $(r, y_2, q), (s, y_2, q) \in \delta$ (see Fig. 10). This implies that $\{r, s\} \times \{y_2 y_1\} \times \{r, s\} \subseteq \delta$. Thus, defining $r_\lambda := r$, $s_\lambda := s$, $v_\lambda := y_2 y_1$ ($\lambda = 1, \ldots, d$) and $u_\lambda := y_2 y_1$ ($\lambda = 2, \ldots, d$), $M$ complies with ($\text{IDA}_d$).

*Case 2*: $M$ does not comply with (EDA).

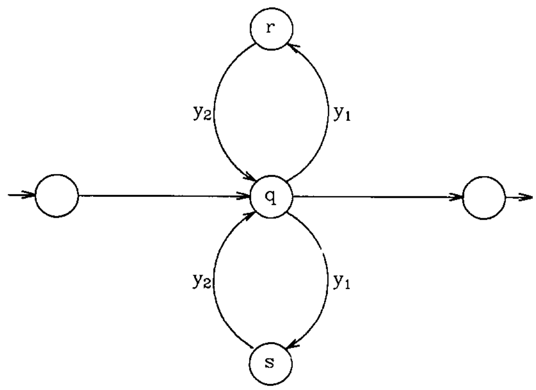



Fig. 10

According to Lemma 2.3, there is a chain NFA $M' \subseteq M$ such that $\deg(M') \geqslant d$. Obviously, $M'$ does not comply with (EDA). Lemma 4.6 implies that $M'$ complies with ($\mathrm{IDA}_d$). Hence, $M$ complies with ($\mathrm{IDA}_d$), too.

(ii) $\Rightarrow$ (i): Let $M$ comply with ($\mathrm{IDA}_d$). Let $r_1, s_1, \ldots, r_d, s_d \in Q$ and $v_1, u_2, v_2, \ldots, u_d, v_d \in \Sigma^*$ be selected according to that criterion. Then, $u_1, w \in \Sigma^*$ exist so that for all $i \in \mathbb{N}$ $\mathrm{da}_M(\prod_{\lambda=1}^{d}(u_\lambda v_\lambda^i)\cdot w) \geqslant \mathrm{da}_M(r_1, v_1^i \prod_{\lambda=2}^{d}(u_\lambda v_\lambda^i), s_d) \geqslant i^d$. Assume that $\deg(M) \leqslant d-1$. Then, there is a polynomial $h \in \mathbb{N}_0[X]$ of degree $d-1$ such that for all $i \in \mathbb{N}$ the following holds: $i^d \leqslant \mathrm{da}_M(\prod_{\lambda=1}^{d}(u_\lambda v_\lambda^i)\cdot w) \leqslant h(\sum_{\lambda=1}^{d}(|u_\lambda| + |v_\lambda|\cdot i) + |w|)$. (Contradiction!) Hence, $\deg(M) \geqslant d$. $\square$

## 5. The maximal finite degree of ambiguity

Let $\Phi$ be a class of NFAs, and let $n \in \mathbb{N}$. We define

$$\mathrm{da}(\Phi, n) := \sup(\mathbb{N}_0 \cap \{\mathrm{da}(M) \mid M \in \Phi, M \text{ has } n \text{ states}\}).$$

According to Theorem 2.1, $\mathrm{da}(\Phi, n)$ is at most $5^{n/2} \cdot n^n$. Thus, $\mathrm{da}(\Phi, n)$ denotes the maximal finite degree of ambiguity of an NFA in $\Phi$ with $n$ states. In this section we deal with the following problem. Given a class $\Phi$ of NFAs and $n \in \mathbb{N}$, determine $\mathrm{da}(\Phi, n)$ or find out lower and upper bounds.

We consider the following classes of NFAs: the class $\Phi_0$ of all NFAs, the class $\Phi_{1S}$ of all NFAs with one input symbol, the class $\Phi_{FL}$ of all NFAs recognizing a finite language, the class $\Phi_C$ of all chain NFAs, the class $\Phi_{C2}$ of all chain NFAs having 2 strong components, the class $\Phi_{CU}$ of all chain NFAs $M$ having only unitary strong components (i.e. "$\underset{M}{\leftrightarrow}$" is the equality relation), and the class $\Phi_{CU,d}$ of all NFAs in $\Phi_{CU}$ with at most $d$ input symbols ($d = 2, 3, 4, 5$).

Our results are summarized in the following theorem.

**Theorem 5.1.** *Let $\Phi$ be a class of* NFAs, *and let $n \in \mathbb{N}$. Then, lower and upper bounds for $\mathrm{da}(\Phi, n)$ hold true as indicated in Table* 1.

Note that in Lines 1–4 of Table 1 $\mathrm{da}(\Phi, n)$ is of order $2^{\Theta(n)}$ while in Lines 5–10 $\mathrm{da}(\Phi, n)$ is of order at least $2^{\Omega(n)}$ and at most $2^{O(n \cdot \log_2 n)}$. In the light of the latter group of results we want to formulate the following question: Where is $\mathrm{da}(\Phi_0, n)$ situated in the range between $2^{\Theta(n)}$ and $2^{\Theta(n \cdot \log_2 n)}$? By Lemma 2.3 we know that $\mathrm{da}(\Phi_C, n) \leqslant \mathrm{da}(\Phi_0, n) \leqslant 5^{n/2} \cdot \mathrm{da}(\Phi_C, n)$. Thus, it seems reasonable to consider the above question for $\Phi_C$ rather than for $\Phi_0$. But, even for $\Phi_{CU,3}$ we do not know the answer. Nevertheless, we conjecture that $\mathrm{da}(\Phi_C, n)$ (and, hence, also $\mathrm{da}(\Phi_0, n)$) is of order $2^{\Theta(n)}$.

**Proof of Theorem 5.1 (summary).**

Line 1: [20, Theorem 5.1].

Line 2: [20, Theorem 5.2].

Line 3: Lemma 5.2 and assertion ($*$) in the proof of Lemma 2.5 (for $k = 2$).

Table 1

| Line | Class $\Phi$ | Lower bound | Upper bound |
|---|---|---|---|
| 1 | $\Phi_{1S}$ = NFAs with one input symbol | $2^{n-1}$ | $2^{n-1}$ |
| 2 | $\Phi_{FL}$ = NFAs recognizing a finite language | $\binom{n}{\lfloor (n+1)/2 \rfloor}$ | $\binom{n}{\lfloor (n+1)/2 \rfloor}$ |
| 3 | $\Phi_{C2}$ = chain NFAs having 2 strong components | if $n \geqslant 2$: $2^{n-2}$ | $2^{n-1}$ |
| 4 | $\Phi_{CU,2}$ = NFAs in $\Phi_{CU}$ with at most 2 input symbols | if $n = 1 \bmod 6$: $2^{0.5283 \cdot (n-1)}$ | $2^{1.5284 \cdot (n-1)}$ |
| 5 | $\Phi_{CU,3}$ = NFAs in $\Phi_{CU}$ with at most 3 input symbols | $2^{n-1}$ | $n^n$ |
| 6 | $\Phi_{CU,4}$ = NFAs in $\Phi_{CU}$ with at most 4 input symbols | if $n = 1 \bmod 3$: $2^{n + \log_2 n - 4.5850}$ | $n^n$ |
| 7 | $\Phi_{CU,5}$ = NFAs in $\Phi_{CU}$ with at most 5 input symbols | if $n = 0 \bmod 64$: $2^{1.0221 \cdot n}$ | $n^n$ |
| 8 | $\Phi_{CU}$ = chain NFAs having only unitary strong components | if $n = 0 \bmod 64$: $2^{1.0221 \cdot n}$ | $n^n$ |
| 9 | $\Phi_C$ = chain NFAs | if $n = 0 \bmod 64$: $2^{1.0221 \cdot n}$ | $n^n$ |
| 10 | $\Phi_0$ = all NFAs | if $n = 0 \bmod 64$: $2^{1.0221 \cdot n}$ | $2^{1.1610 \cdot n} \cdot n^n$ |

Line 4: Lemmas 5.6 and 5.7.
Line 5: [20, Lemma 5.5] and Lemma 2.5.
Line 6: [20, Lemma 5.6] and Lemma 2.5.
Line 7: [20, Lemma 5.7] and Lemma 2.5.
Lines 8–9: Line 7 and Lemma 2.5.
Line 10: Line 7 and Theorem 2.1. □

The next lemma proves the lower bound in Line 3 of Table 1.

**Lemma 5.2.** *For all $n_1, n_2 \in \mathbb{N}$ there is a trim* NFA $M := M_{n_1, n_2} \in \Phi_{C2}$ *with $n_1 + n_2$ states and $n_1 + n_2 + 2$ input symbols such that the following assertions are true*:

(i) *The two strong components of $M$ have $n_1$ and $n_2$ states, respectively.*

(ii) $2^{n_1 + n_2 - 2} \leqslant \mathrm{da}(M_{n_1, n_2}) < \infty$.

**Proof.** Let $n_1, n_2 \in \mathbb{N}$. We construct $M := M_{n_1, n_2} = (Q, \Sigma, \delta, Q_\mathrm{I}, Q_\mathrm{F})$:

$$Q := Q_1 \dot{\cup} Q_2,\ Q_1 := \{p_1, \ldots, p_{n_1}\},\ Q_2 := \{q_1, \ldots, q_{n_2}\},\ Q_\mathrm{I} := \{p_{n_1}\},\ Q_\mathrm{F} := \{q_{n_2}\},$$
$$\Sigma := \{i \in \mathbb{Z} \mid -(n_2 - 1) \leqslant i \leqslant n_1 - 1\} \dot{\cup} \{a_1, a_2, a_3\},$$
$$\delta := \bigcup_{a \in \Sigma} \delta(a),$$

where

- $\delta(i) := \{(p_j, i, p_j) \mid i+2 \leqslant j \leqslant n_1\} \cup \{p_{i+1}\} \times \{i\} \times \{p_1, \ldots, p_i\} \cup \{(q_j, i, q_j) \mid 1 \leqslant j \leqslant n_2\}$ $(i \in [n_1 - 1])$,
- $\delta(0) := \{(p_j, 0, p_j) \mid 2 \leqslant j \leqslant n_1\} \cup \{(p_1, 0, q_1)\} \cup \{(q_j, 0, q_j) \mid 1 \leqslant j \leqslant n_2\}$,
- $\delta(-i) := \{p_1\} \times \{-i\} \times \{q_1, p_1, \ldots, p_{n_1}\} \dot{\cup} \{q_1, \ldots, q_i\} \times \{-i\} \times \{q_{i+1}\}$ $\cup \{(q_j, -i, q_j) \mid i+2 \leqslant j \leqslant n_2\}$ $(i \in [n_2 - 1])$,
- $\delta(a_1) := \{p_{n_1}\} \times \{a_1\} \times Q_1$, $\delta(a_2) := Q_2 \times \{a_2\} \times \{q_{n_2}\}$, $\delta(a_3) := \{(q_{n_2}, a_3, q_1)\}$.

(See Fig. 11 for the definition of $\delta$ in the case $n_1 = 5$ and $n_2 = 4$.)

$M$ is a trim chain NFA having the strong components $Q_1$ and $Q_2$. Thus, it also meets the assertion (i). $M$ is planned to recognize a suitable input word by "counting" through almost all sets of its states.

**Claim 5.3.** $M$ is finitely ambiguous.

Assume that $\mathrm{da}(M)$ is infinite. Then, by Theorem 3.1, $M$ complies with (IDA). By inspection of $M$, we observe that for all $(p', x, q') \in Q_i \times \Sigma^* \times Q_i$, $\mathrm{da}_M(p', x, q')$ is at most 1 $(i = 1, 2)$. Therefore, according to (IDA), $p \in Q_1$, $q \in Q_2$ and $v \in \Sigma^*$ exist such that $(p, v, p), (p, v, q), (q, v, q) \in \delta$. From this follows that $q = q_1$ and $v \in \{0, \ldots, n_1 - 1\}^*$. This implies that $(p, v, p) \notin \delta$ or $(p, v, q) \notin \delta$. (Contradiction!)

Definition of $\delta$ ($n_1 = 5$, $n_2 = 4$):

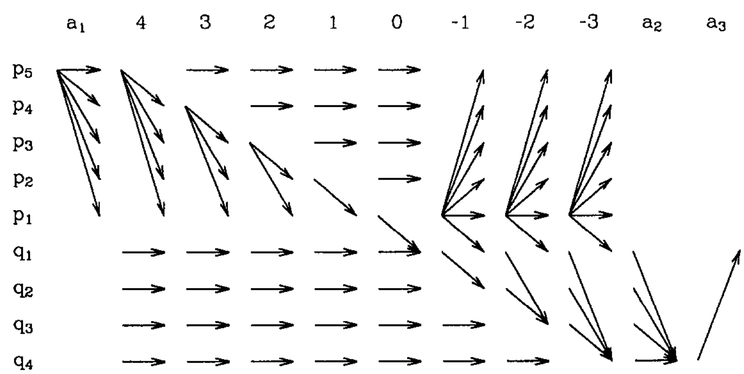



Fig. 11

$G_M(y)$ ($n_1=4$, $n_2=3$):

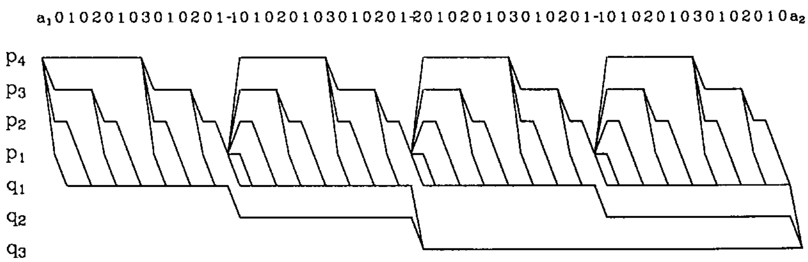



Fig. 12

**Claim 5.4.** $\mathrm{da}(M) \geqslant 2^{n_1+n_2-2}$.

We construct the words $y_0^{(1)} := \varepsilon$, $y_i^{(1)} := y_{i-1}^{(1)} 0 i y_{i-1}^{(1)}$ $(i=1,\ldots,n_1-1)$, $y_0^{(2)} := y_{n_1-1}^{(1)}$, $y_i^{(2)} := y_{i-1}^{(2)}(-i) y_{i-1}^{(2)}$ $(i=1,\ldots,n_2-1)$, and $y := a_1 y_{n_2-1}^{(2)} 0 a_2$. Counting selected accepting paths for $y$ in $M$ it is easy to show that $\mathrm{da}_M(y) \geqslant 2^{n_1+n_2-2}$ (see Fig. 12 for the graph $G_M(y)$ in the case $n_1=4$ and $n_2=3$).

From Claims 5.3 and 5.4 it follows that $M$ meets the assertion (ii). □

We add to the proof of Lemma 5.2 that the NFA $M_{n_1,n_2}$ turns out to be quite useful in order to construct finite-valued finite transducers [20, Chapter 12; 21, Section 4] and finite monoids of matrices with entries in $\mathbb{N}_0$ [23, Section 2].

Let $M=(Q,\Sigma,\delta,Q_\mathrm{I},Q_\mathrm{F})$ be a trim NFA in $\Phi_{\mathrm{CU}}$ with $n$ states. According to the definition of $\Phi_{\mathrm{CU}}$, there is an order $q_1,\ldots,q_n$ of the states of $M$ such that $Q_\mathrm{I}=\{q_1\}$, $Q_\mathrm{F}=\{q_n\}$ and $\delta \subseteq \bigcup_{i=1}^{n}\{q_i\}\times\Sigma\times\{q_i\} \cup \bigcup_{i=1}^{n-1}\{q_i\}\times\Sigma\times\{q_{i+1}\}$. Thus, the criterion (IDA) introduced in Section 3 reads for $M$ as follows:

$$\text{(CU-IDA):}\qquad \exists 1\leqslant i<j\leqslant n\ \exists a_i,\ldots,a_{j-1}\in\Sigma\ \forall\kappa=i,\ldots,j-1:$$

$$(q_i,a_\kappa,q_i)\in\delta\ \ \&\ \ (q_\kappa,a_\kappa,q_{\kappa+1})\in\delta\ \ \&\ \ (q_j,a_\kappa,q_j)\in\delta.$$

Together with Theorem 3.1, this implies Lemma 5.5.

**Lemma 5.5.** *Let $M$ be a trim* NFA *in $\Phi_{\mathrm{CU}}$ as above. $M$ is finitely ambiguous if and only if the following assertion is true*:

$$\text{(CU-FDA):}\qquad \forall 1\leqslant i<j\leqslant n\ \exists i\leqslant\kappa=\kappa(i,j)<j\ \forall a\in\Sigma:$$

$$(q_i,a,q_i)\in\delta\ \ \&\ \ (q_j,a,q_j)\in\delta \Rightarrow (q_\kappa,a,q_{\kappa+1})\notin\delta.$$

Using Lemma 5.5 we are able to establish the bounds in Line 4 of Table 1 with Lemmas 5.6 and 5.7.

**Lemma 5.6.** *For each $n = 1 \bmod 6$ there is a trim* NFA $M_n \in \Phi_{\mathrm{CU},2}$ *with n states such that* $2^{0.5283 \cdot (n-1)} < 3^{(n-1)/3} \leqslant \mathrm{da}(M_n) < \infty$.

**Proof.** Let $n = 1 \bmod 6$. We construct $M_n = (Q, \Sigma, \delta, Q_{\mathrm{I}}, Q_{\mathrm{F}})$:

$$Q := \{q_1, \ldots, q_n\},\ \Sigma := \{1,2\},\ Q_{\mathrm{I}} := \{q_1\},\ Q_{\mathrm{F}} := \{q_n\},$$

$$\begin{aligned}\delta := &\{(q_{6i+1}, 1, q_{6i+1}) \mid 0 \leqslant i \leqslant (n-1)/6\}\\ &\cup \{(q_i, 1, q_{i+1}) \mid 1 \leqslant i \leqslant n-1,\ i \neq 3 \bmod 6\}\\ &\cup \{(q_{6i+3}, 2, q_{6i+3}) \mid 0 \leqslant i \leqslant (n-7)/6\}\\ &\cup \{(q_i, 2, q_{i+1}) \mid 1 \leqslant i \leqslant n-1,\ i \neq 0 \bmod 6\}.\end{aligned}$$

We define for all $1 \leqslant i < j \leqslant n$,

$$\kappa(i,j) := \begin{cases} i+2 & \text{if } i,j = 1 \bmod 6,\\ i+3 & \text{if } i,j = 3 \bmod 6,\\ i & \text{otherwise.}\end{cases}$$

Using this definition it can be easily checked that $M_n$ is a trim NFA in $\Phi_{\mathrm{CU},2}$ which complies with (CU-FDA). Consider $y := (1^2 2^5 1^3)^{(n-1)/6} \in \Sigma^*$. Counting selected accepting paths for $y$ in $M_n$ it is easy to show that $\mathrm{da}_{M_n}(y) \geqslant 9^{(n-1)/6}$ (see Fig. 13). In conclusion, we know by Lemma 5.5 that

$$3^{(n-1)/3} \leqslant \mathrm{da}_{M_n}(y) \leqslant \mathrm{da}(M_n) < \infty. \quad \square$$

Definition of $\delta$ and $G_{M_n}(y)$ ($n = 13$):

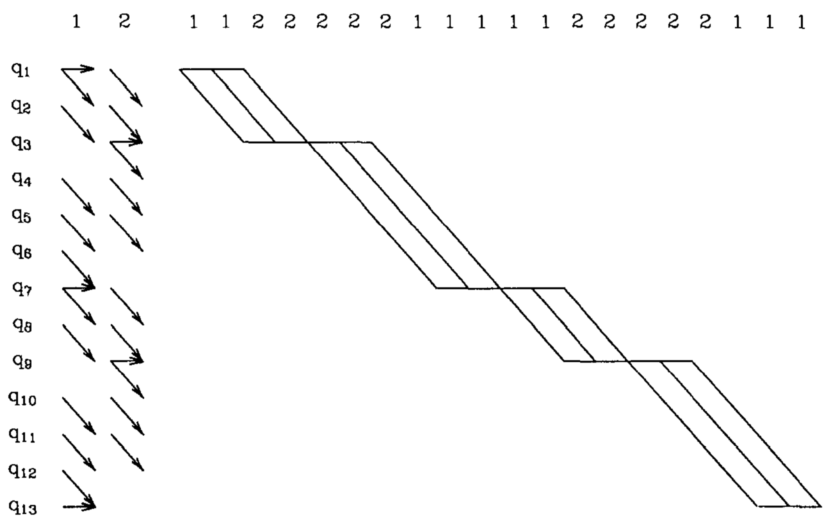



Fig. 13

**Lemma 5.7.** *Let $M$ be a finitely ambiguous* NFA *in $\Phi_{\mathrm{CU},2}$ with $n$ states. Then, the degree of ambiguity of $M$ is at most $2^{n-1}\cdot 3^{(n-1)/3}$.*

**Proof.** We assume that $M=(Q,\Sigma,\delta,Q_{\mathrm{I}},Q_{\mathrm{F}})$ is trim and corresponds with the notation used in Lemma 5.5. Thus, there is an order $q_1,\ldots,q_n$ of the states of $M$ such that $Q_{\mathrm{I}}=\{q_1\}$, $Q_{\mathrm{F}}=\{q_n\}$ and $\delta\subseteq\bigcup_{i=1}^{n}\{q_i\}\times\Sigma\times\{q_i\}\cup\bigcup_{i=1}^{n-1}\{q_i\}\times\Sigma\times\{q_{i+1}\}$. Let $\Sigma=\{1,2\}$.

For each $\boldsymbol{a}=(a_1,\ldots,a_{n-1})\in\Sigma^{n-1}$ we construct the NFA $M^{(\boldsymbol{a})}=(Q,\Sigma,\delta^{(\boldsymbol{a})},Q_{\mathrm{I}},Q_{\mathrm{F}})\subseteq M$:

$$\delta^{(\boldsymbol{a})}:=\delta\cap\left(\bigcup_{i=1}^{n}\{q_i\}\times\Sigma\times\{q_i\}\cup\bigcup_{i=1}^{n-1}\{(q_i,a_i,q_{i+1})\}\right).$$

Note that $(q_1,a_1,q_2),\ldots,(q_{n-1},a_{n-1},q_n)$ are the only possible bridges of $M^{(\boldsymbol{a})}$. We observe for all $x\in\Sigma^*$ that

$$\mathrm{da}_M(x)=\sum_{\boldsymbol{a}\in\Sigma^{n-1}}\mathrm{da}_{M^{(\boldsymbol{a})}}(x).$$

This implies that

$$\mathrm{da}(M)\leqslant\sum_{\boldsymbol{a}\in\Sigma^{n-1}}\mathrm{da}(M^{(\boldsymbol{a})})<\infty.$$

Therefore, the lemma follows from assertion $(*)$.

$(*)$ Let $M$ be an NFA as above. Let $a_1,\ldots,a_{n-1}\in\Sigma$ so that $(q_1,a_1,q_2),\ldots,(q_{n-1},a_{n-1},q_n)$ are the only bridges of $M$. Then, $\mathrm{da}(M)$ is at most $3^{(n-1)/3}$.

We prove $(*)$ by induction on $n$. If $n=1$, then $M$ is unambiguous, i.e. $\mathrm{da}(M)=1$.

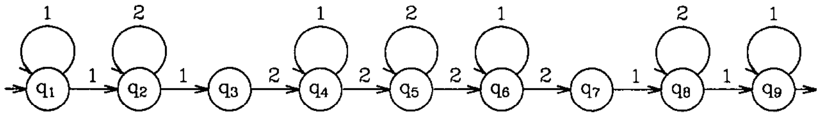
Let $n>1$. By Lemma 5.5, $M$ complies with (CU-FDA). Assume that $\{q_1\}\times\Sigma\times\{q_1\}\cup\{q_n\}\times\Sigma\times\{q_n\}\subseteq\delta$. Then, taking $\kappa:=\kappa(1,n)\in[n-1]$ as in (CU-FDA), we observe that $\delta\cap\{q_\kappa\}\times\Sigma\times\{q_{\kappa+1}\}=\emptyset$. (Contradiction!) We can w.l.o.g. assume that $(q_1,2,q_1)\notin\delta$, $(q_1,1,q_1)\in\delta$, and that $a_1=1$. Define $l:=\max\{2\leqslant i\leqslant n\,|\,a_1=\cdots=a_{i-1}=1\}$. Thus, $a_1=\cdots=a_{l-1}=1$ and either $l=n$ or $a_l=2$. Since $(q_1,1,q_1)\in\delta$ and $M$ complies with (CU-FDA), we observe for all $i=2,\ldots,l$ that $(q_i,1,q_i)\notin\delta$ (see Fig. 14).

Let $x\in\Sigma^*$. Let $t\in\mathbb{N}_0$ and $y\in\{\varepsilon\}\cup\{2\}\cdot\Sigma^*$ so that $x=1^t y$. From the induction hypothesis it follows that

$$\mathrm{da}_M(x)=\sum_{i=2}^{l}\mathrm{da}_M(q_1,1^t,q_i)\cdot\mathrm{da}_M(q_i,y,q_n)\leqslant\sum_{i=2}^{l}3^{(n-(l-1)-1)/3}$$

$$=(l-1)\cdot 3^{(n-l)/3}\leqslant 3^{(l-1)/3}\cdot 3^{(n-l)/3}=3^{(n-1)/3}.$$

Note that $j\leqslant 3^{j/3}$ $(j\in\mathbb{N})$. Thus, $\mathrm{da}(M)$ is at most $3^{(n-1)/3}$.




$l = 3$ and $n = 9$

Fig. 14

Let us remark that for each $n = 1 \bmod 4$ there is an NFA $M_n$ as in $(*)$ such that $2^{(n-1)/4} \leqslant \mathrm{da}(M_n) < \infty$. $M_9$ is presented in Fig. 14. In fact, $4 \leqslant \mathrm{da}_{M_9}(yy^R) \leqslant \mathrm{da}(M_9) < \infty$, where $y := 112122$. $\square$

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