Lecture 1

Weighted automata basic definitions

Finite automata

Definition

A finite automaton is $\mathcal{A} = (Q, \Sigma, T, I, F)$, where:

- Q is a finite set of states
- Σ is a finite alphabet
- $T \subseteq Q \times \Sigma \times Q$ is a finite set of transitions
- $I, F \subseteq Q$ are the sets of initial and final states

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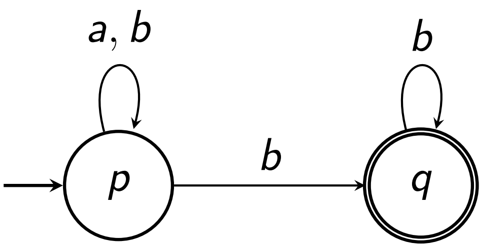
Example:



- $Q = \{p, q\}$
- $\Sigma = \{a, b\}$
- $T = \{(p, a, p), (p, b, p), (p, b, q), (q, b, q)\}$
- $I = \{p\}$, $F = \{q\}$

Finite automata runs

Let $\mathcal{A} = (Q, \Sigma, T, I, F)$

A run of $\mathcal{A}$ on $w = a_1 \dots a_n \in \Sigma^*$ is $\rho = t_1 \dots t_n$, where:

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$L(\mathcal{A}) = \{w \mid [\![\mathcal{A}]\!](w) = true\}$

Automata counting things

What about $\mathcal{A} : \Sigma^* \to$ numbers, $\mathbb{N}$?, $\mathbb{Q}$?

How many a's are there in the word?

What is the probability of acceptance?

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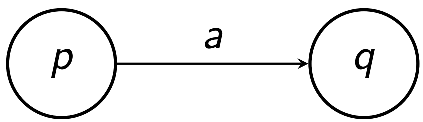
Add values on transitions



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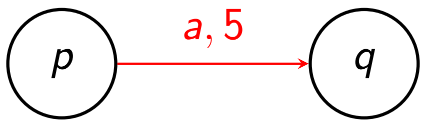

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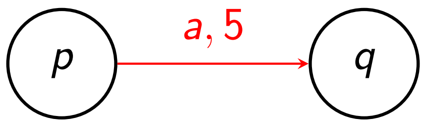

To discuss what numbers first we will describe the semiring structure in the following slides

Commutative semirings

$\mathbb{S}(\oplus, \odot, \mathbb{0}, \mathbb{1})$ a set $\mathbb{S}$ with two operations and axioms

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$\mathbb{S}(\oplus, \odot, \mathbb{0}, \mathbb{1})$ a set $\mathbb{S}$ with two operations and axioms

1. $(\mathbb{S}, \oplus)$ is a commutative monoid with identity $\mathbb{0}$

- $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
- $\mathbb{0} \oplus a = a \oplus \mathbb{0} = a$
- $a \oplus b = b \oplus a$

2. $(\mathbb{S}, \odot)$ is a commutative monoid with identity $\mathbb{1}$

- $(a \odot b) \odot c = a \odot (b \odot c)$
- $\mathbb{1} \odot a = a \odot \mathbb{1} = a$
- $a \odot b = b \odot a$

3. Distributivity

- $a \odot (b \oplus c) = a \odot b \oplus a \odot c$

4. Annihilation

- $\mathbb{0} \odot a = a \odot \mathbb{0} = \mathbb{0}$

(Commutative) semirings examples

- Rings like $(\mathbb{Q}, +, \cdot, 0, 1)$
- Natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$

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Tropical semirings

- $(\mathbb{N}_{+\infty}, \min, +, \infty, 0)$, where $\mathbb{N}_{+\infty} = \mathbb{N} \cup \{+\infty\}$
- $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$, where $\mathbb{N}_{+\infty} = \mathbb{N} \cup \{-\infty\}$

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Note: $\oplus = \min$, $\odot = +$, $\mathbb{0} = +\infty, \mathbb{1} = 0$

Axioms work:

$n \oplus \mathbb{0} = n \quad$ becomes $\quad \min(n, +\infty) = n$

$n \odot \mathbb{1} = n \quad$ becomes $\quad n + 0 = n$

Weighted automata

Definition

A weighted automaton over a semiring $\mathbb{S}(\oplus, \odot, \mathbb{0}, \mathbb{1})$ is $\mathcal{A} = (Q, \Sigma, T, I, F)$:

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For every $p, q \in Q$ and $a \in \Sigma$ we assume there is at most one $(p, a, s, q) \in T$

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Remark

For every $p, q \in Q$ and $a \in \Sigma$ we assume there is at most one $(p, a, s, q) \in T$

So ignoring $\mathbb{S}$ in T

and identifying I (and F) with $I' = \{q \mid I(q) \neq \mathbb{0}\}$

we get a finite automaton

Weighted automata runs and output

Given $\mathcal{A} = (Q, \Sigma, T, I, F)$ over $\mathbb{S}(\oplus, \odot, \mathbb{0}, \mathbb{1})$

A run of $\mathcal{A}$ on $w = a_1 \ldots a_n \in \Sigma^*$ is $\rho = t_1 \ldots t_n$, where:

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For every $\rho = t_1 \dots t_n \in R_w$: $val(\rho) = I(p_0) \odot \bigodot_{i=1}^{n} val(t_i) \odot F(q_n)$

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Then $[\![\mathcal{A}]\!](w) = \bigoplus_{\rho \in R_w} val(\rho)$ $\qquad [\![\mathcal{A}]\!](\epsilon) = \bigoplus_{q \in Q} I(q) \odot F(q)$

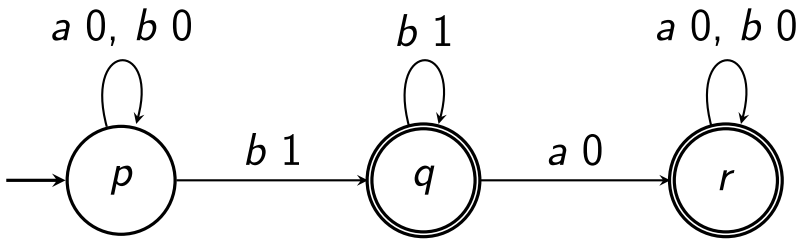
Weighted automata examples

Fix the semiring $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$

Weighted automata examples

Fix the semiring $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$

- Longest block of b's



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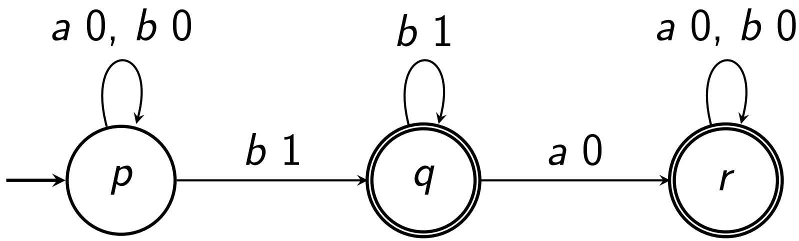

$T = \{(p,a,0,p),(p,b,0,p),(p,b,1,q),(q,b,1,q),(q,a,0,r),\ldots\}$

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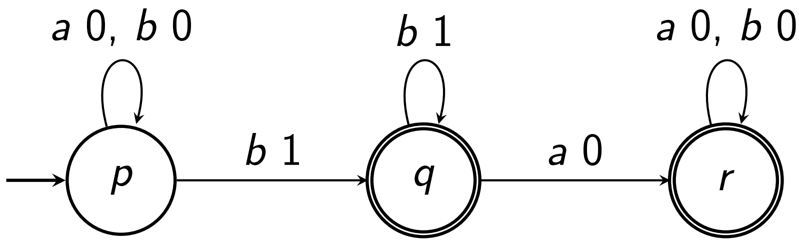

$$T = \{(p, a, 0, p), (p, b, 0, p), (p, b, 1, q), (q, b, 1, q), (q, a, 0, r), \ldots\}$$

Remark

Usually $I, F : Q \to \{\mathbb{0}, \mathbb{1}\} = \{+\infty, 0\}$. Then initial state means the value of I is $\mathbb{1}$ and $\mathbb{0}$ otherwise. Here, $I(p) = 0$, $I(q) = +\infty$ and $I(r) = +\infty$. Similarly with accepting states and F.

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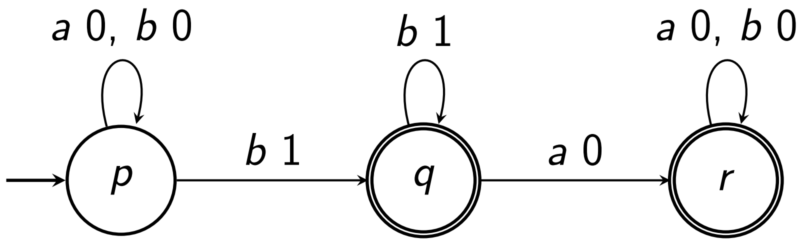

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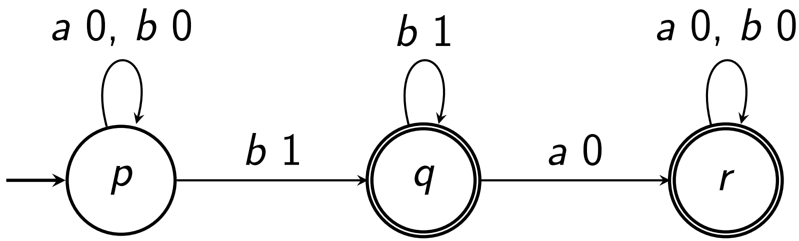

- Let $w = bbab$

  All runs starting in q or r have value $-\infty + \ldots + = -\infty$

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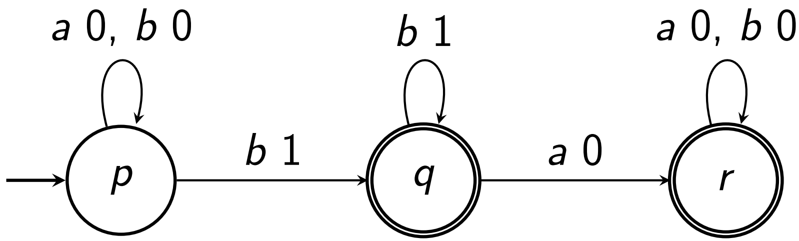

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There are three other runs (skipping 0's from I and F)

$1 + 1 + 0 + 0 = 2, \quad 0 + 1 + 0 + 0 = 1, \quad 0 + 0 + 0 + 1 = 2$

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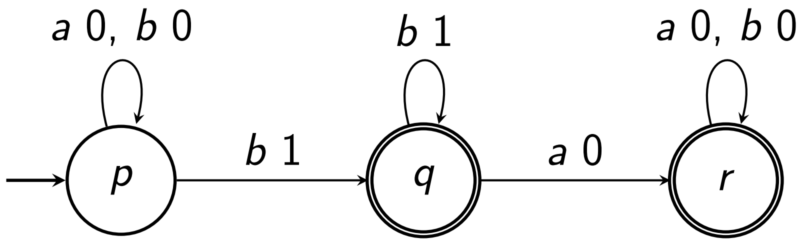

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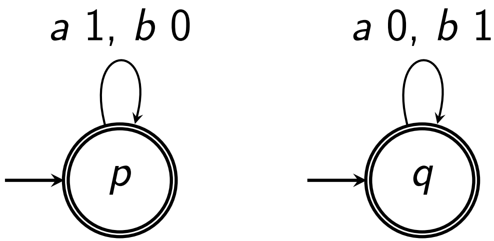
$1 + 1 + 0 + 0 = 2, \quad 0 + 1 + 0 + 0 = 1, \quad 0 + 0 + 0 + 1 = 2$

$[\![\mathcal{A}]\!]\,(bbab) = \max\{2, 1, 1, -\infty\} = 2$

Weighted automata examples

Fix the semiring $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$

- Maximum of number of a's and number of b's



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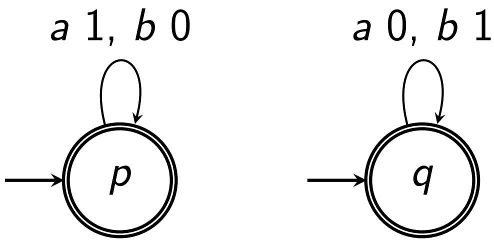

There are always two runs. Consider *bbab*

$0 + 0 + 1 + 0 = 1$ and $1 + 1 + 0 + 1 = 3$

Output: $\max\{1, 3\} = 3$

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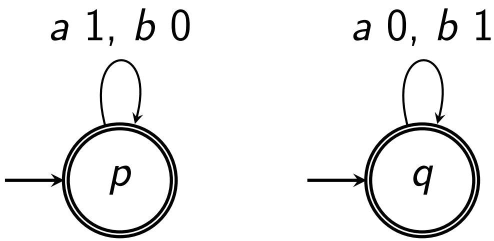

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$0 \cdot 0 \cdot 1 \cdot 0 = 0$ and $1 \cdot 1 \cdot 0 \cdot 1 = 0$

Output: $0 + 0 = 0$

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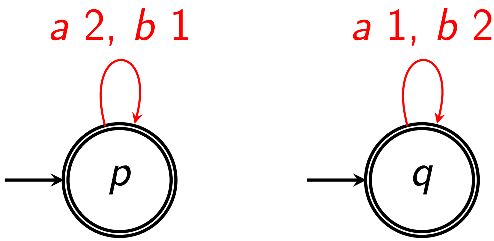

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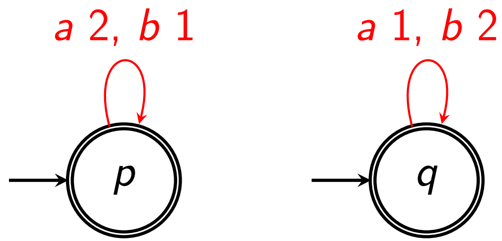


There are always two runs. Consider *bbab*

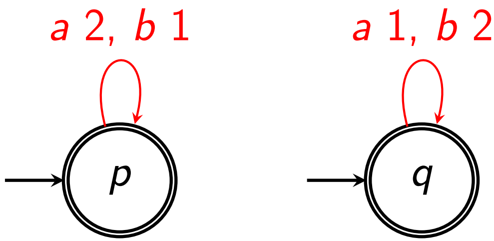
$1 \cdot 1 \cdot 2 \cdot 1 = 2$ and $2 \cdot 2 \cdot 1 \cdot 2 = 8$

Output: $2 + 8 = 10$

Weighted automata examples

Change the semiring to $(\mathbb{Q}, +, \cdot, 0, 1)$

This is $[\![\mathcal{A}]\!](w) = 2^{\#_a(w)} + 2^{\#_b(w)}$



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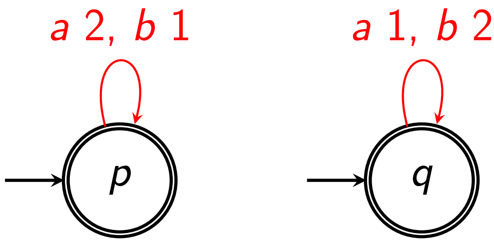

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It is important to write the semiring of the weighted automaton

Weighted automata over the boolean semiring

Consider the semiring $(\{0,1\}, \vee, \wedge, 0, 1)$ (0 is false, 1 is true)

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- Definitions as expected

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- Weighted automata over this semiring are finite automata

  Initial, final states are states such that $I(q) = 1$ and $F(q) = 1$

  Transitions in finite automata are transitions such that $val(t) = 1$

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  Then $val(\rho) = 1 \wedge 1 \wedge 1 \ldots \wedge 1 = 1$ if ρ is accepting

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  The output is

$$[\![\mathcal{A}]\!](w) = \bigvee val(\rho)$$

Weighted automata different definition

Definition

A weighted automaton $\mathcal{A}$ over $\mathbb{S}(\oplus, \odot, \mathbb{0}, \mathbb{1})$ is $(d, \Sigma, \{M_a\}_{a\in\Sigma}, I, F)$, where:

- $d \in \mathbb{N}$ is the dimension;
- Σ is a finite alphabet;
- every M_a is a $d \times d$ matrix over $\mathbb{S}$;
- I and F are the initial and the final vector in $\mathbb{S}^d$.

Weighted automata different definition

Definition

A weighted automaton $\mathcal{A}$ over $\mathbb{S}(\oplus, \odot, \mathbb{0}, \mathbb{1})$ is $(d, \Sigma, \{M_a\}_{a\in\Sigma}, I, F)$, where:

- $d \in \mathbb{N}$ is the dimension;
- Σ is a finite alphabet;
- every M_a is a $d \times d$ matrix over $\mathbb{S}$;
- I and F are the initial and the final vector in $\mathbb{S}^d$.

$[\![\mathcal{A}]\!] : \Sigma^* \to \mathbb{Q}$

$$\boxed{[\![\mathcal{A}]\!]\,(a_1 a_2 \ldots a_n) = I^{\mathsf{T}} \odot M_{a_1} M_{a_2} \ldots M_{a_n} \odot F}$$

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Remark

It makes sense to multiply matrices over any semiring. Over $\mathbb{N}(\max, +)$:

$$\begin{pmatrix} 0 & -\infty \\ -\infty & 1 \end{pmatrix} \begin{pmatrix} 1 & -\infty \\ -\infty & 0 \end{pmatrix} = \begin{pmatrix} \max(0+1, -\infty + -\infty) & \max(0 + -\infty, -\infty + 0) \\ \max(-\infty + 1, 1 + -\infty) & \max(1+0, -\infty + -\infty) \end{pmatrix}$$

Definition equivalence

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  $Q = \{1, \ldots, d\}$

  Vectors $F, I \in \mathbb{S}^d$ are the same thing as a functions $F, I : Q \to \mathbb{S}$

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Theorem

$[\![\mathcal{A}]\!](w) = [\![\mathcal{A}']\!](w)$ for all $w \in \Sigma^*$.

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Proof.

$$[\![\mathcal{A}]\!](\epsilon) = I^{\mathsf{T}} \odot F, \qquad [\![\mathcal{A}']\!](\epsilon) = \bigoplus_{i=1}^{d} I(i) \odot F(i)$$

Induction for $|w| > 0$

Definition

$R_w^{p,q}$ is the set of runs in $\mathcal{A}'$ from state p to state q over w

For every $\rho = t_1 \dots t_n$ we denote by $trans(\rho) = val(t_1) \odot \dots \odot val(t_n)$

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if $|w| = 1$ then there is only one run from p to q

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if $|w| > 1$ then write $w = av$ for $v \in \Sigma^+$ and $a \in \Sigma$

Induction continued

Notation $t_i = (p_i, a_i, s_i, q_i) \in T$, where $q_i = p_{i+1}$

Then $R_w^{p,q} = \{t_1 t_2 \ldots t_n \mid p_1 = p, t_2 \ldots t_n \in R_v^{p_2,q}\}$

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The lemma follows from the definition of matrix multiplication

For any matrices A, B in dimensions d we have

$$AB[p, q] = \bigoplus_{i \in \{1, \dots, d\}} A[p, i] B[i, q]$$

Final step

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$\blacksquare$

The opposite translation on tutorials