# Lecture 3

Ambiguity of automata

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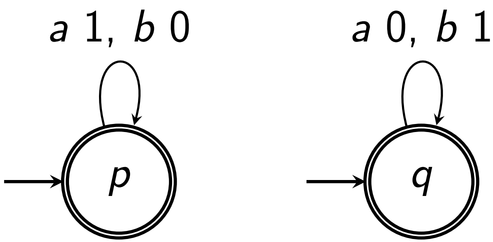

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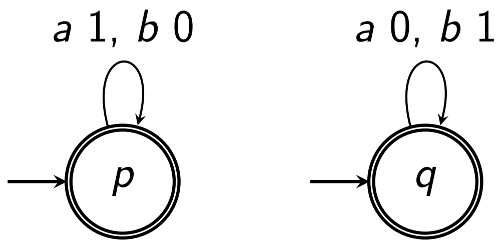

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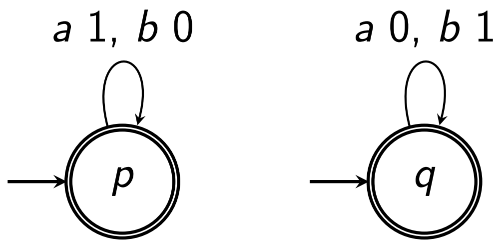


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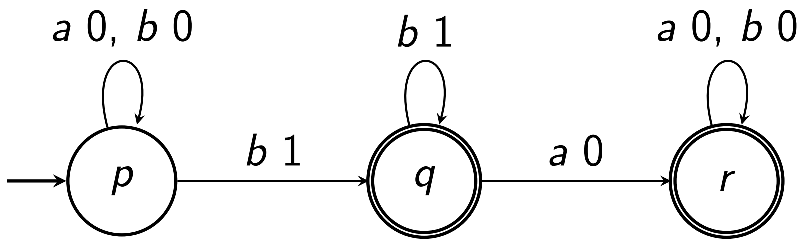

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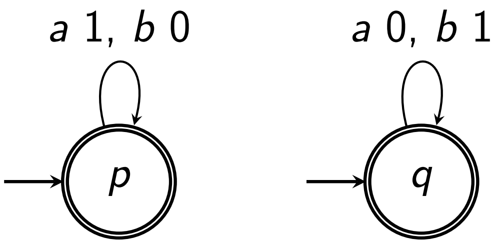


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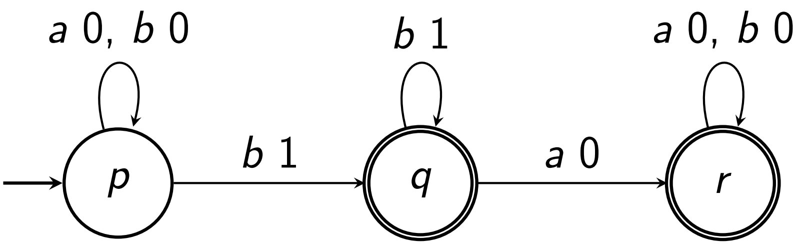
  $\mathcal{O}(|w|)$ runs



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## Hierarchy of classes for weighted automata

Weighted automata (WA)

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Deterministic WA

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An automaton $\mathcal{A}$ is trimmed if for every $q \in Q$ there is an initial state $p \in I$ and a final state $r \in F$ s.t. there is a run from $p$ to $q$ and a run from $q$ to $r$.

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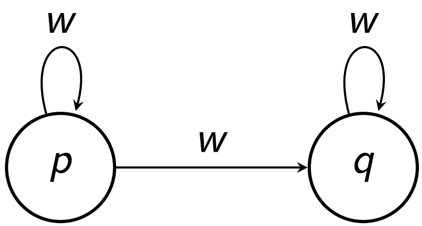

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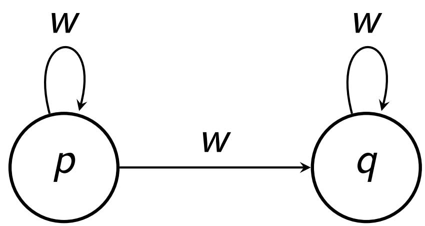


**Proof. (2) $\implies$ (1)**

There is $a \in I$ and $v_a$ s.t. $a \xrightarrow{v_a} p$ and $b \in F$ and $v_b$ s.t. $q \xrightarrow{v_b} b$

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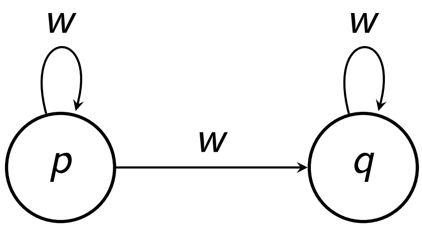


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Then $|Acc(v_a w^n v_b)| \geqslant n - 1$.

# Not finitely ambiguous $\implies$ pattern (1)

- A strongly connected component is $Q_i \subseteq Q$ s.t. for all $p, q \in Q$ there are $v_1, v_2$: $p \xrightarrow{v_1} q$ and $q \xrightarrow{v_2} p$

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$Q = Q_1 \cup Q_2 \ldots \cup Q_m$, where $Q_i$ are strongly connected components. Moreover, if $p \in Q_i$ and $q \in Q_j$ and there is a word $w$ s.t. $p \xrightarrow{w} q$ then $i \leqslant j$.

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So sets $G$ is a DAG and $Q_a$ can be topologically sorted

■

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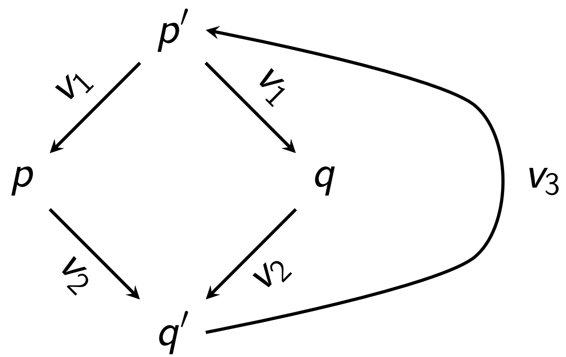

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- Let $w = v_2 v_3 v_1$

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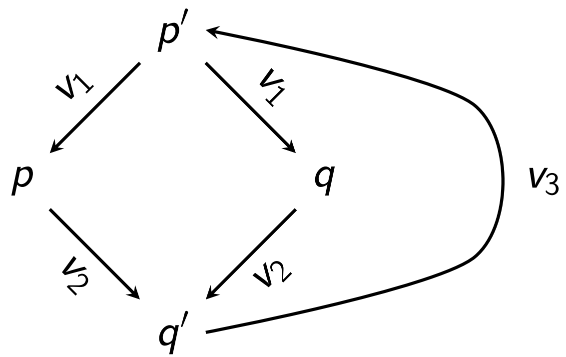

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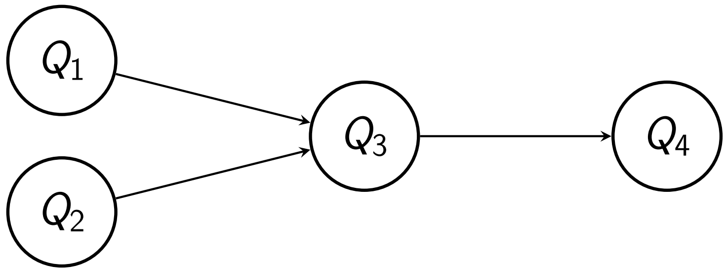

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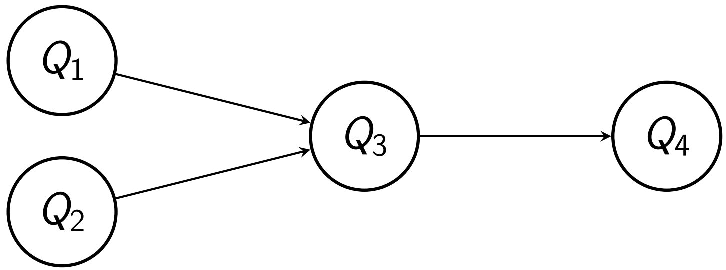


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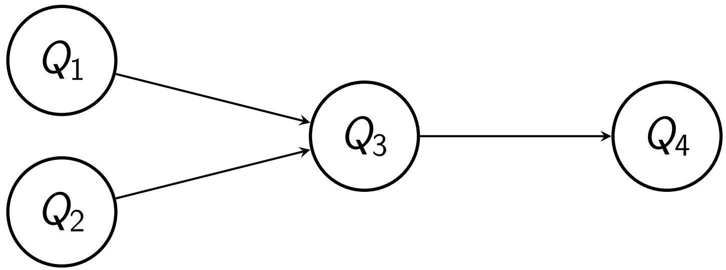


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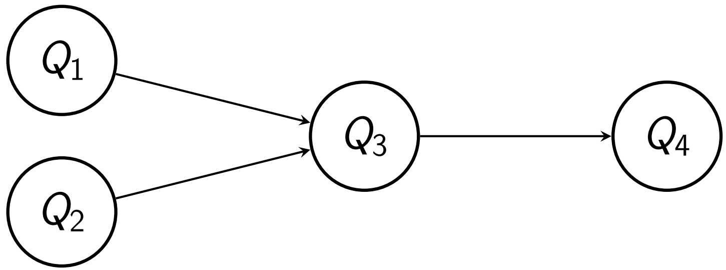


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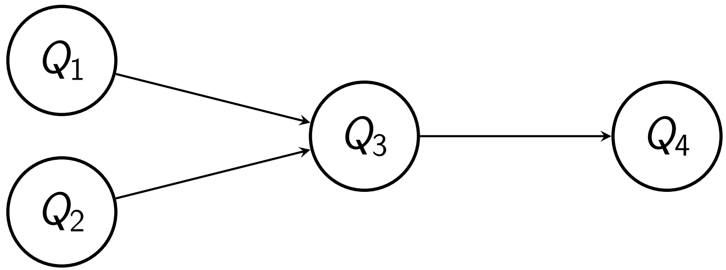


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- Note: the number of such $M$ is bounded by $2^{|Q|}$.

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Let $x = x_1 \ldots x_s \in \Sigma^*$. The graph $G_M(x) = (V, E)$ is defined as

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$E = \{(p, j-1) \to (q, j) \mid p \xrightarrow{x_j} q\}$

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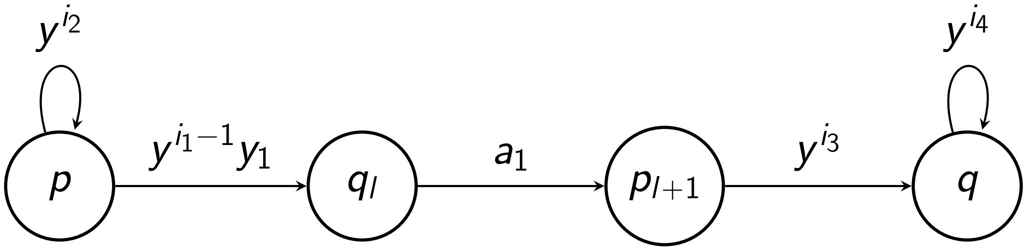
  For some $i_1$, $i_2$ we get $r_{i_1} = r_{i_1+i_2} = p$
- Similarly a sequence $s_i$: $\quad s_0 = p_{l+1}$ and $s_{i-1} \xrightarrow{y} s_i$, $\quad$ so $s_{i_3} = s_{i_3+i_4} = q$

# Not finitely ambiguous $\implies$ pattern (7)

$y = y_1 a_1$




- $p \xrightarrow{y^{i_2}} p, \quad p \xrightarrow{y^{i_1-1}y_1} q_l, \quad q_l \xrightarrow{a_1} p_{l+1}, \quad p_{l+1} \xrightarrow{y^{i_3}} q, \quad q \xrightarrow{y^{i_4}} q$

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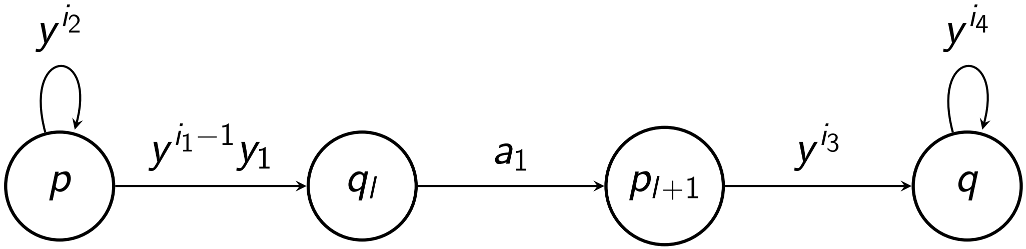


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- $p \neq q$, let $w = y^{j \cdot i_2 \cdot i_4}$

  $p \xrightarrow{w} p,\ p \xrightarrow{w} q,\ q \xrightarrow{w} q$

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**Theorem** (Weber, Seidl 1991)

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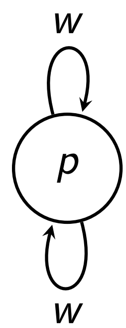

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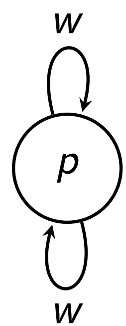


**Proof.** **(2) $\implies$ (1)**

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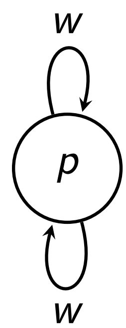


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Then $|Acc(v_a w^n v_b)| \geqslant 2^n$.

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$\blacksquare$

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**Lemma** (tutorials)

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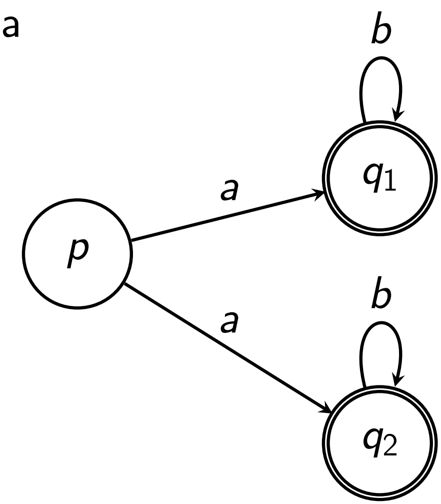

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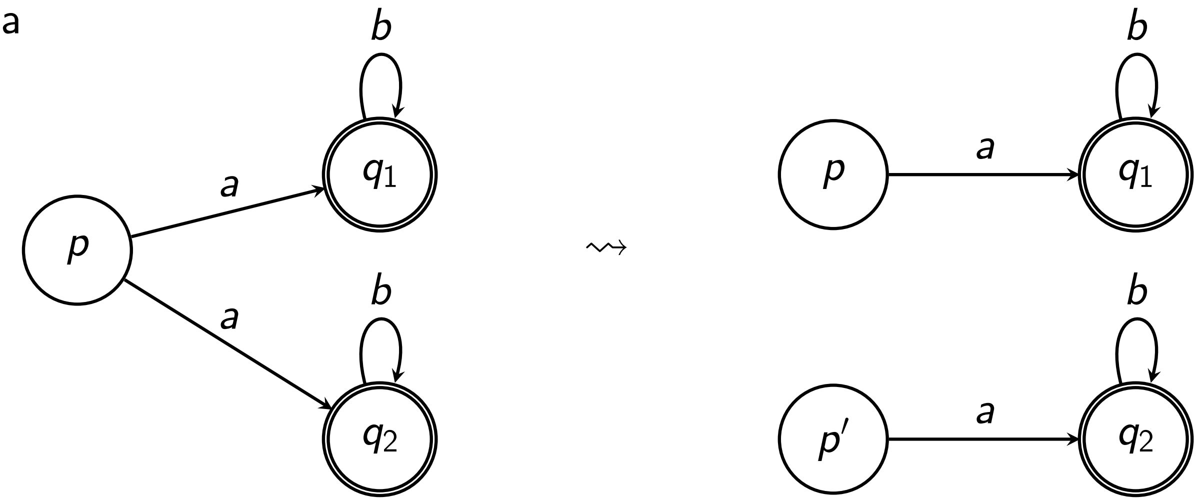

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- $\mathcal{B}$ keeps track of all active runs in $\mathcal{A}$

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- Then for every transition in $\mathcal{A}$ we update the states in $\mathcal{B}$

  Given a partial function $f : \{1, 2, \ldots, kn\} \to Q$

  we get $g : \{1, 2, \ldots, kn\} \to Q$

## Construction of the union of unambiguous automata (2)

- $\mathcal{B}$ states are partial functions $f : \{1, 2, \ldots, kn\} \to Q$

  The size of the domain is always the number of runs

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  Given a partial function $f : \{1, 2, \ldots, kn\} \to Q$

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- Then $\mathcal{A} \times \mathcal{B}$ has the same accepting runs as $\mathcal{A}$ but with extra information

  Accepting states etc are when they are accepting in the $\mathcal{A}$ component

# Construction of the union of unambiguous automata (3)

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- The final automata are divided into $kn$ unambiguous automata
  Restricting the accepting states to accepting in $\mathcal{A}$
  and $i \in \{1, \ldots, kn\}$ in the final component