Lecture 4

Ambiguity for 1-letter alphabets

Hierarchy of classes for weighted automata

- The inclusions are strict for $(\mathbb{Q}, +, \cdot, 0, 1)$ even over 1-letter alphabet

Weighted automata (WA)

$\cup\!|$

Polynomially ambiguous WA

$\cup\!|$

Finitely ambiguous WA

$\cup\!|$

Unambiguous WA

$\cup\!|$

Deterministic WA

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- Most inclusions: tutorials
- We focus on the most difficult one

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Polynomially ambiguous sequences over $\mathbb{Q}$

Fix the semiring $(\mathbb{Q}, +, \cdot, 0, 1)$

Theorem (Barloy et. al 2019)

Polynomial ambiguous WA over 1-letter alphabet $=$ linear recursive sequences with eigenvalues that are roots of rational numbers.

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Corollary

Fibonacci sequence is not definable by polynomially ambiguous WA.

Formal power series

For example $u_n = 1$

$$U(x) = \sum_{n=0}^{\infty} u_n \cdot x^n \; = \; \sum_{n=0}^{\infty} 1 \cdot x^n \; = \; \frac{1}{1-x}$$

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where $P(x)$ of degree at most k

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  $p^*(x) = x^k p(\frac{1}{x})$, so λ root of $p(x)$ iff $\frac{1}{\lambda}$ root of $p^*(x)$

Polynomially ambiguous $\implies$ eigenvalues are roots of rationals (1)

- Fix $\mathcal{A}$ polynomially ambiguous (trimmed) $\Sigma = \{a\}$

  How do strongly connected components look like?

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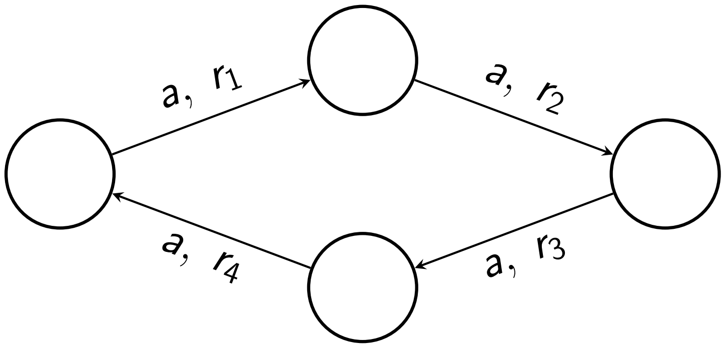

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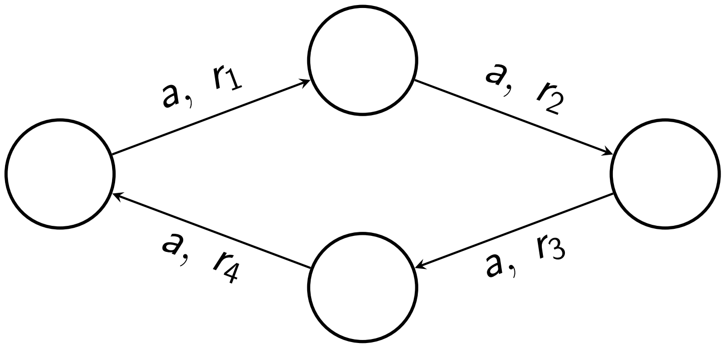
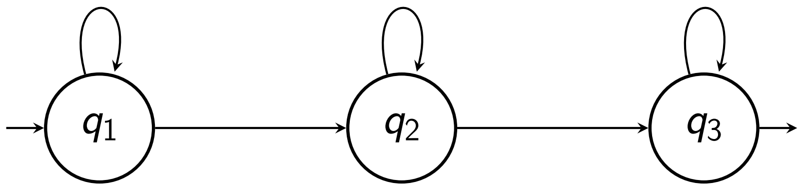


- Chained loops

  q_i loops in connected components

  Initial state q_1, final state q_3



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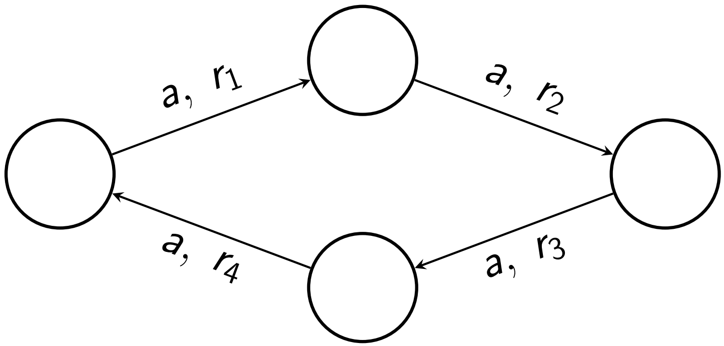

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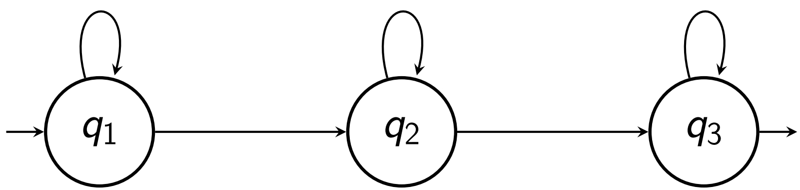

- By previous lecture $\mathcal{A}$ is equivalent to a union of chained loops

Polynomially ambiguous $\implies$ eigenvalues are roots of rationals (2)

Lemma

The formal series induced by a chained loop with 1 strongly connected component is $\frac{\alpha}{1-\lambda x^l}$ where α is the initial weight, λ is the product of weights and l number of states.

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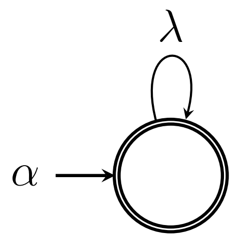

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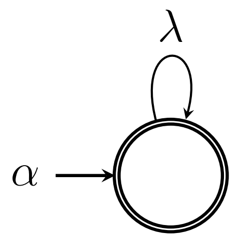

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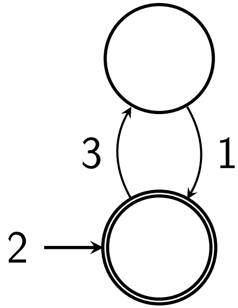
Definition

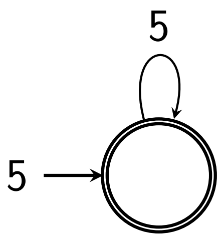
$\mathcal{A}_1$ and $\mathcal{A}_2$ chained loops. Their concatenation $\mathcal{A} = \mathcal{A}_1\mathcal{A}_2$ is the union of states, where: the initial state is from $\mathcal{A}_1$; the final state is from $\mathcal{A}_2$ and merging the output of $\mathcal{A}_1$ and input of $\mathcal{A}_2$ into a transition

Polynomially ambiguous $\implies$ eigenvalues are roots of rationals (3)

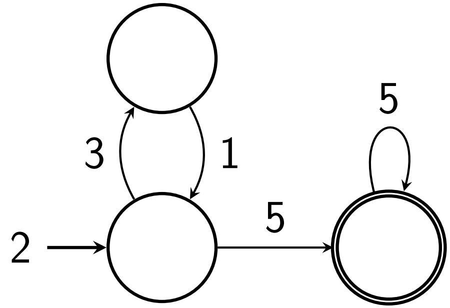
$\mathcal{A}_1$



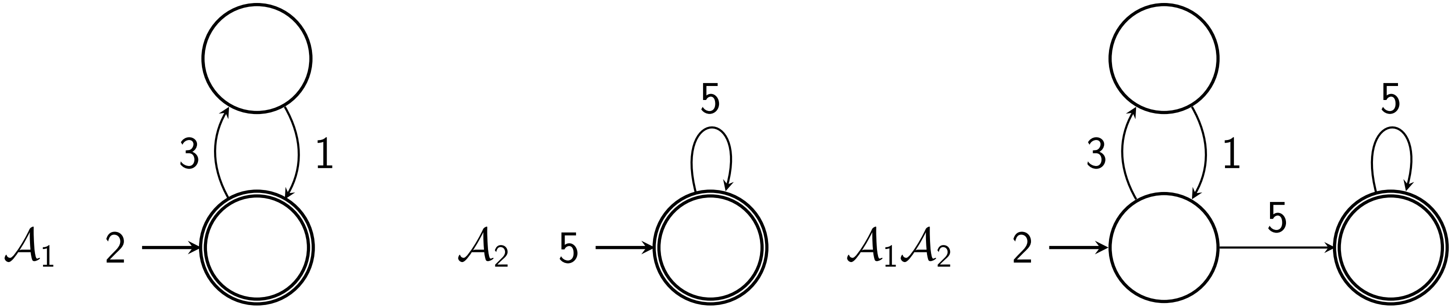
$\mathcal{A}_2$



$\mathcal{A}_1\mathcal{A}_2$



Polynomially ambiguous $\implies$ eigenvalues are roots of rationals (3)



Lemma

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ chained loops and let S_1 and S_2 be their formal power series.

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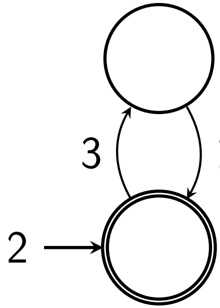
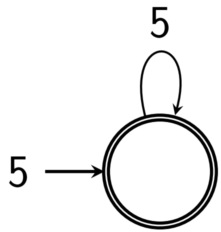
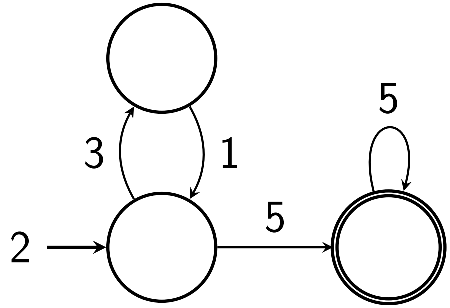
$\mathcal{A}_1$ 

$\mathcal{A}_2$ 

$\mathcal{A}_1\mathcal{A}_2$ 

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$$\mathcal{A}_1\mathcal{A}_2(n) = \sum_{i=1}^{n} \mathcal{A}_1(i-1)\mathcal{A}_2(n-i) = \sum_{i=0}^{n-1} \mathcal{A}_1(i)\mathcal{A}_2(n-i-1)$$

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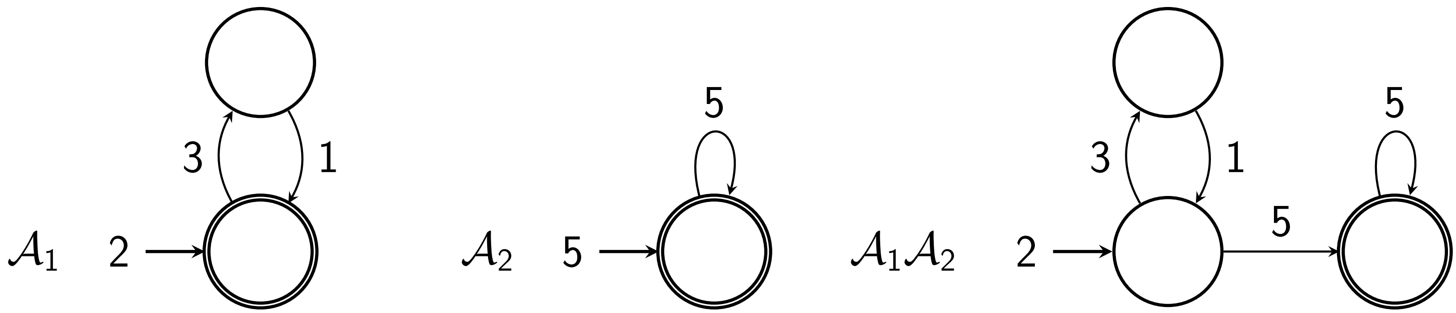

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$$\sum_{n=0}^{\infty} u_n x^n \cdot \sum_{n=0}^{\infty} v_{n-1} x^n \quad \text{is equal to} \quad \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} u_i v_{n-i-1} \right) x^n$$

$\blacksquare$

Polynomially ambiguous $\implies$ eigenvalues are roots of rationals (4)

- Let $\mathcal{A}_1$ and $\mathcal{A}_2$ chained loops and let S_1 and S_2 be their formal power series. The formal power series of $\mathcal{A}_1$ union $\mathcal{A}_2$ is $S_1 + S_2$.

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- By all lemmas the power series of a polynomial ambiguous automaton is $\frac{P(x)}{Q(x)}$ where $Q(x) = \prod_i \left(1 - \lambda_i x^{l_i}\right)$ and $\lambda_i \in \mathbb{Q}$

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- If P and Q are coprime then Q is reciprocal of the characteristic polynomial

  So roots of the characteristic polynomial are roots of $\frac{1}{\lambda_i}$

■

Eigenvalues are roots of rationals $\implies$ polynomially ambiguous (1)

Consider a sequence with eigenvalues that are root of rationals

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Consider a sequence with eigenvalues that are root of rationals

- So formal power series $\frac{P(x)}{Q(x)}$, where $Q(x) = \prod_i \left(1 - \lambda_i x^{l_i}\right)$
- Polynomials with rational coefficients $\mathbb{Q}[x]$ are Euclidean
  $\implies$ if $P, Q \in \mathbb{Q}[x]$ don't share roots then there are $R_1, R_2 \in \mathbb{Q}$ s.t.
  $R_1 P + R_2 Q = 1$

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- Let $P = \left(1 - \lambda x^{l}\right)^{s}$, $Q = \left(1 - \lambda' x^{l'}\right)^{t'}$ for $\lambda \neq \lambda'$
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  There are R_1, R_2 s.t. $R_1 P + R_2 Q = 1$
- Then $\frac{A}{PQ} = \frac{A(R_1 P + R_2 Q)}{PQ} = \frac{AR_2}{P} + \frac{AR_1}{Q}$

Eigenvalues are roots of rationals $\implies$ polynomially ambiguous (2)

By induction $$\frac{P(x)}{Q(x)} = \sum_i \frac{A_i}{(1-\lambda_i x^{l_i})^{t_i}}$$

Eigenvalues are roots of rationals $\implies$ polynomially ambiguous (2)

By induction $\dfrac{P(x)}{Q(x)} = \sum_i \dfrac{A_i}{\left(1 - \lambda_i x^{l_i}\right)^{t_i}}$

- Polynomial ambiguous WA are closed under sum and shifts

It remains to prove that $\frac{1}{\left(1-\lambda_i x^l\right)^t}$ is polynomial ambiguous

Eigenvalues are roots of rationals $\implies$ polynomially ambiguous (2)

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$\blacksquare$

Remark

Here we rely on $\mathbb{Q}$. For example Fibonacci over $\mathbb{R}$ is finitely ambiguous because $F_n = (\frac{1+\sqrt{5}}{2\sqrt{5}})^n + (\frac{1-\sqrt{5}}{2\sqrt{5}})^n$

The $(\mathbb{N}_{+\infty}, \min, +, \infty, 0)$ and $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$ semirings

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Fix a 1-letter alphabet and $(\mathbb{N}_{+\infty}, \min, +, \infty, 0)$ or $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$ semiring. Then Weighted automata are equivalent to unambiguous weighted automata.

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Two steps:

1. From WA to finitely ambiguous WA
2. From finitely ambiguous WA to unambiguous WA

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We do (1) now, (2) on tutorials

From WA to finitely ambiguous WA (1)

Let $\mathcal{A}$ over with 1-letter alphabet over $(\mathbb{N}_{-\infty}, \mathsf{max}, +, -\infty, 0)$ (trimmed) and let $|Q| = n$

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Definition

Let $\rho = q_0, \ldots, q_m$ be a run and let $P = \{q_0, \ldots, q_m\}$ (the set of states without repetitions). A simple cycle decomposition of ρ is obtained as follows: (1) mark one state from P in ρ (these cannot be removed); (2) from the beginning if you see a simple cycle remove it modulo (1); (3) stop when nothing can be removed.

From WA to finitely ambiguous WA (2)

$q \qquad p \qquad p \qquad p \qquad r \qquad p \qquad q$

From WA to finitely ambiguous WA (2)

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- We denote $val(\rho, f)$ the value of any run obtained from it

From WA to finitely ambiguous WA (3)

Lemma

Fix a word a^m. For every run ρ there is a simple cycle decomposition on (δ, f) such that: (1) at most one simple cycle occurs more than n times; (2) $val(\delta, f) \geqslant val(\rho)$

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- Notice that k copies of σ' and k' copies of σ have the same length (bounded by n^2)

From WA to finitely ambiguous WA (4)

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- So for every maximal run there is a simple cycle decomposition with the same weight, where at most one cycle occurs many times

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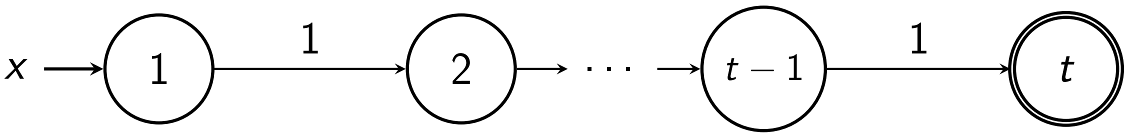

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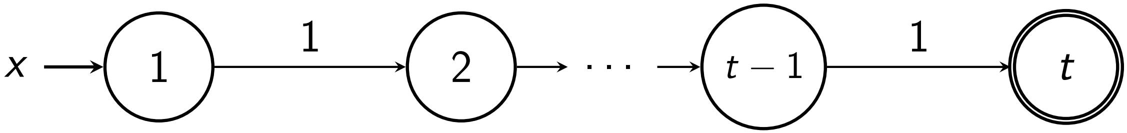
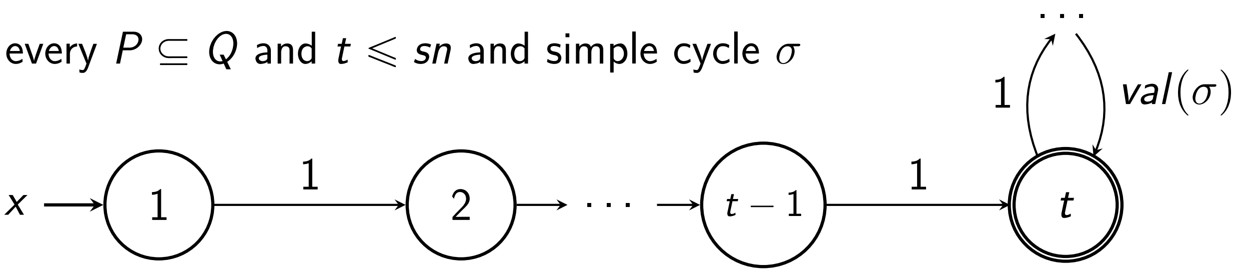


where x is the output for a^t

- For every $P \subseteq Q$ and $t \leqslant sn$ and simple cycle σ



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- To prove that the ambiguity hierarchy is strict for $(\mathbb{N}_{+\infty}, \min, +, \infty, 0)$ and $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$ we need at least 2 letters
- On tutorials you've already seen examples (maximal number of a's and b's)
- Next week more details