Lecture 5

Ambiguity for the max plus semiring

Hierarchy of classes for weighted automata

- The inclusions are strict for $(\mathbb{N}_{+\infty}, \min, +, \infty, 0)$ and $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$

$$\begin{array}{c} \text{Weighted automata (WA)} \\ \cup\!\mid \\ \text{Polynomially ambiguous WA} \\ \cup\!\mid \\ \text{Finitely ambiguous WA} \\ \cup\!\mid \\ \text{Unambiguous WA} \\ \cup\!\mid \\ \text{Deterministic WA} \end{array}$$

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Weighted automata (WA)

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Polynomially ambiguous WA

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Finitely ambiguous WA

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Unambiguous WA

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Deterministic WA

- We focus on $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$
  unambiguous, finitely ambiguous and polynomially ambiguous

Pumping lemmas

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Example: $aab \cdot \underline{bb} \cdot ba$ refines $aa \cdot bbbb \cdot a$

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Theorem (M. and Riveros 2018)

Let $f : \Sigma^* \to \mathbb{N} \cup \{-\infty\}$ be definable by unambiguous WA over $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$. There exists N such that for all words of the form $u \cdot v \cdot w \in \Sigma^*$ with $|v| \geqslant N$, $v \neq \epsilon$, there exists a refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ of $u \cdot \underline{v} \cdot w$ such that one of the conditions holds:

(1) $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) = f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geqslant N$.

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  $f(a^{(N+1)^2} \cdot b^N \cdot \epsilon) = (N+1)^2$, refining $a^{(N+1)^2} \cdot b^N$ we get $a^{(N+1)^2} b^n \cdot \underline{b^m} \cdot b^l$

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- Since $n + mN + l < (N+1)^2$ then (1). But for i big enough (2)

Unambiguous automata (2)

Corollary

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Let $\mathcal{A}$ unambiguous automaton defining f. And let uvw with $v \geqslant N >> 2^{|Q|}$

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Let $\mathcal{A}$ unambiguous automaton defining f. And let uvw with $v \geqslant N >> 2^{|Q|}$

- if $f(uvw) > -\infty$ then there is a unique accepting run on uvw

  Let $q_0, \ldots, q_{|v|}$ be the set of states on v

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- if $f(uvw) = -\infty$ then since there is at most $n = exp(|Q|)$ runs on uvw

  ($\mathcal{A}$ is unambiguous)

Unambiguous automata (3)

We can present all runs on v as sequences

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- If new runs occur then the number of runs for uv^iw would be at least $n + i - 1$

  (contradiction with finite ambiguity)

■

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Let $w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \ldots u_{n-1} \cdot \underline{v_n} \cdot u_n$

A refinement is $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$

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if $v_k = x_k \cdot y_k \cdot z_k$, $u'_k = z_k \cdot u_k \cdot x_{k+1}$; where $z_0 = x_{n+1} = \epsilon$.

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- Let $S \subseteq \{1, \ldots, n\}$

  Then $\underline{y_k}(S, i)$ is y_k^i if $k \in S$ and y_k otherwise

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  By $w(S, i)$ we denote the word

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- Let $f : \Sigma^* \to \mathbb{N} \cup \{-\infty\}$. A refinement is linear if for all S there exists K s.t.

  $f(w(S, i + i)) = K + f(w(S, i))$ for all i big enough

  For linear refinements we denote $\Delta(S) = K$

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  S is decomposable if $\Delta(S) = \sum_{j \in S} \Delta(\{j\})$

Finitely ambiguous automata (2)

Theorem

Let f definable by finitely ambiguous automaton over $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$.

There exists $N \in \mathbb{N}$ such that for every $w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \ldots \underline{v_n} \cdot u_n$, where $n \geqslant N$ and and $|v_i| \geqslant N$ for all i, there exists a linear refinement

$w = x_0 \cdot \underline{y_1} \cdot x_1 \cdot \underline{y_2} \cdot \ldots \underline{y_n} \cdot x_n$

such that for every sequence of pairwise different, non-empty sets $S_1, S_2, \ldots S_k \subseteq \{1, \ldots, n\}$ with $k \geqslant N$, one of the following holds:

(1) exists j s.t. S_j is not decomposable

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  Let $(\underline{b^{N+1}}a)^{N+1}$ and define $S_j = \{j\}$

  Every S_j is decomposable but any $\{j_1, j_2\}$ is not decomposable

Corollary

Finitely ambiguous WA $\subsetneq$ poly-ambiguous WA over $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$

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- **Proof.**

  Suppose f is recognised by $\mathcal{A}$ which is m-ambiguous, $|Q| = r$ and $f(w) > -\infty$

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- Set $N = \max\{r^m, m\} + 1$.

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- Such a refinement is linear

  and $\Delta(S)$ is determined by maximal cycle weights in blocks in S

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  Suppose f is recognised by $\mathcal{A}$ which is m-ambiguous, $|Q| = r$ and $f(w) > -\infty$

- Set $N = \max\{r^m, m\} + 1$.

  The refinement is defined in such a way that every y_i is a cycle

  on all accepting runs

- Such a refinement is linear

  and $\Delta(S)$ is determined by maximal cycle weights in blocks in S

- We will assume there are only accepting runs (to simplify technicalities)

Finitely ambiguous automata (4)

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Let $S \subseteq \{1, \ldots, n\}$ a linear refinement. Then S is decomposable iff for one of the runs ρ the cycle $\rho[j]$ is dominant for all $j \in S$.

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- By lemma $\{k_1, k_2\}$ is decomposable for every $k_1 \in S_{j_1}$ and $k_2 \in S_{j_2}$

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Polynomially ambiguous automata (1)

Let $S_1, \ldots, S_m$ over $\{1, \ldots, n\}$ a *partition* (S_i nonempty, pairwise disjoint)

We say that $S \subseteq \{1, \ldots, n\}$ is a *selection set* if $|S \cap S_i| = 1$ for every i.

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Theorem

Let f poly-ambiguous over $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$. There exist N and a function $\varphi : \mathbb{N} \to \mathbb{N}$ such that for all $w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \ldots u_{n-1} \cdot \underline{v_n} \cdot u_n$, where $|v_i| \geqslant N$ for every $1 \leqslant i \leqslant n$, there exists a linear refinement

$$w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdots u'_{n-1} \cdot \underline{y_n} \cdot u'_n,$$

such that for every partition $\pi = S_1, S_2, \ldots S_m$ of $\{1, \ldots, n\}$ with $m \geqslant \varphi(\max_j(|S_j|))$ one of the following holds:

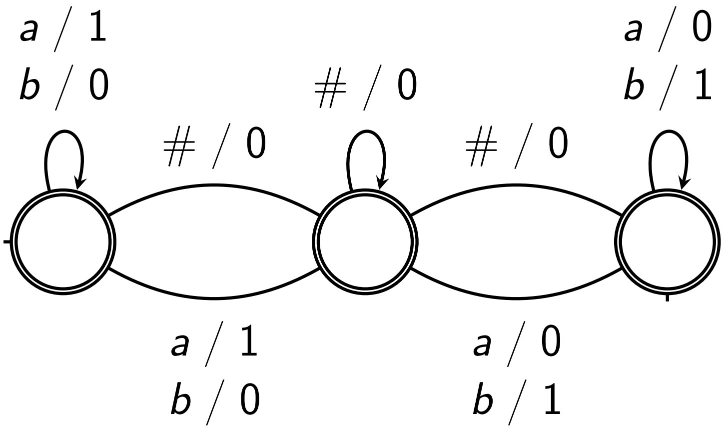
(1) there exists j such that S_j is decomposable;

(2) there exists a selection set S for π such that S is not decomposable.

Polynomially ambiguous automata (2)

Example $w_0\#w_1\#\dots\#w_n$ with $w_i \in \{a,b\}^*$

$f(w) = \sum_{i=0}^{n} \max\{|w_i|_a, |w_i|_b\}$



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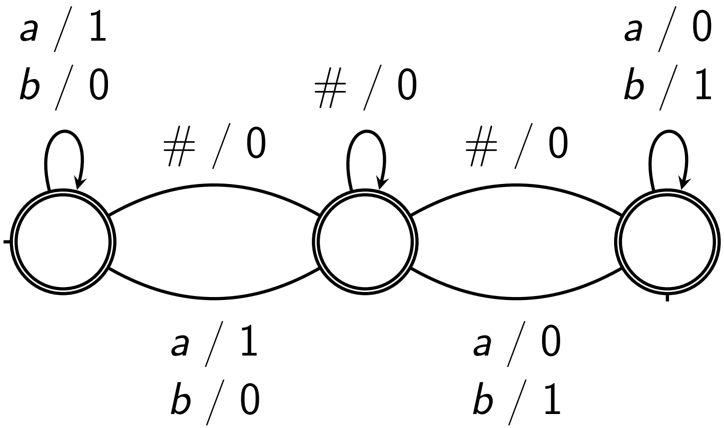

- Fix N and φ from the theorem. Let $m \geqslant \varphi(2)$ consider refinements of $(\underline{a^N}\,\underline{b^N}\#)^m$

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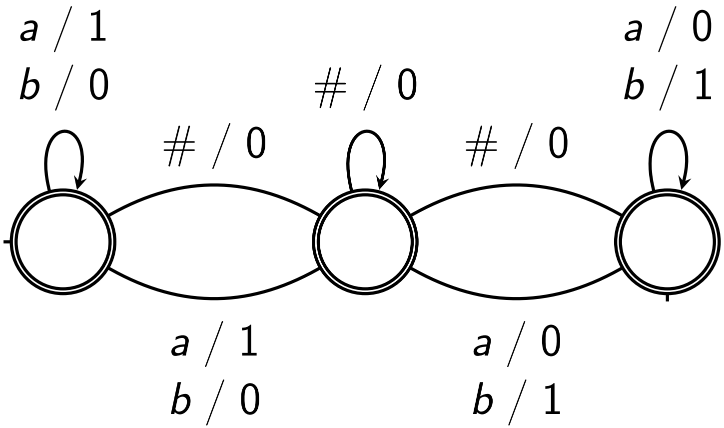

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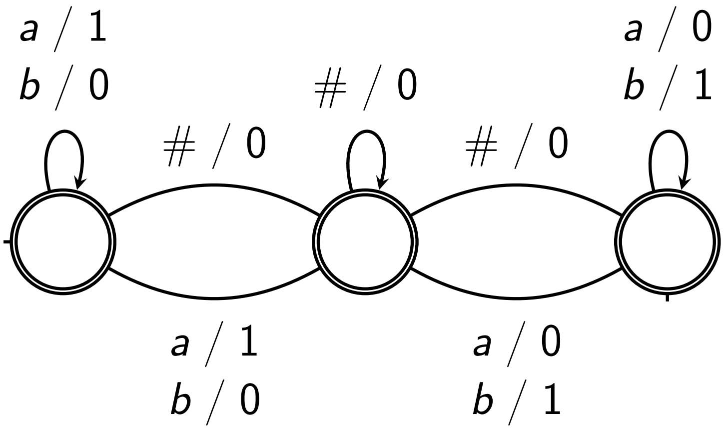

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  But every selection set is decomposable

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Poly-ambiguous WA $\subsetneq$ WA over $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$

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  Long-standing open problem for $(\mathbb{N}_{+\infty}, \min, +, \infty, 0)$ and $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$

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- Partial results: decidable for $(\mathbb{N}_{+\infty}, \min, +, \infty, 0)$ and $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$
  If we assume that $\mathcal{A}$ is unambiguous, finitely ambiguous or poly-ambiguous

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Variants of the classical emptiness problems for finite automata

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Given a linear recursive sequence $u_0, u_1, \ldots$ is there $n \in \mathbb{N}$ s.t. $u_n = 0$

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This is known as the Skolem problem (open for many years)

Next two weeks there will be a result related to this problem

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Theorem (Bertoni 1974)

Fix the semiring $(\mathbb{Q}, +, \cdot, 0, 1)$. The problem if given $\mathcal{A}$ is there a word w such that $[\![\mathcal{A}]\!](w) = 0$ is undecidable.

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Given two morphisms $\varphi_1, \varphi_2 : \Sigma^* \to \{0, 1\}^*$ is there a word w s.t. $\varphi_1(w) = \varphi_2(w)$

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- One can assume that if such a w exists then the last letter of w is 1

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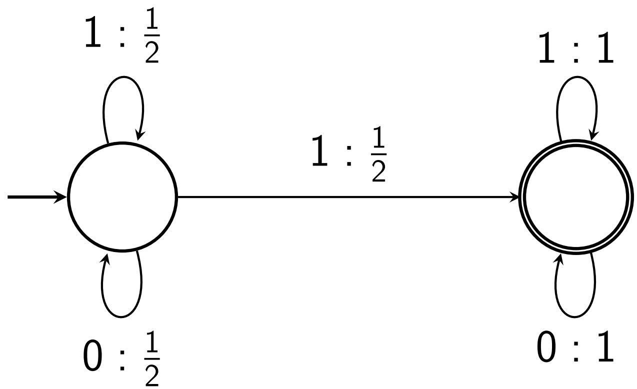

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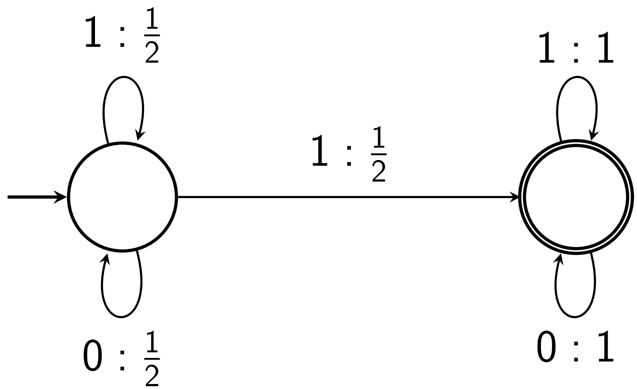
- $\mathcal{A}(a_1 a_2 \ldots a_n) = bin(a_1 \ldots a_n)$
  $= \sum_{i=1}^{n} \frac{a_i}{2^i}$



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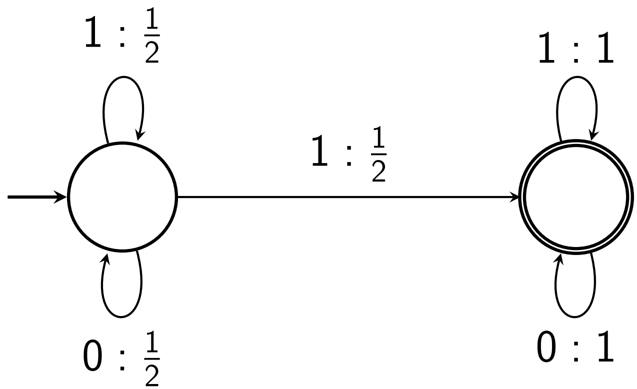

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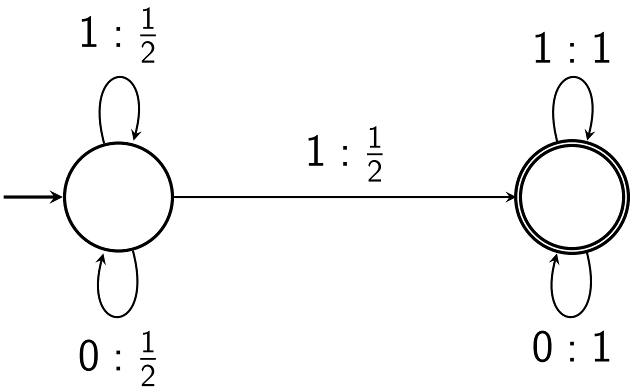

- Notice that (almost) every w has a unique value $[\![\mathcal{A}]\!](w)$ (assuming w ends with 1)

- Given a morphism $\varphi : \Sigma^* \to \{0,1\}^*$ we define $\mathcal{A}_\varphi$ s.t. $[\![\mathcal{A}_\varphi]\!](w) = bin(\varphi(w))$

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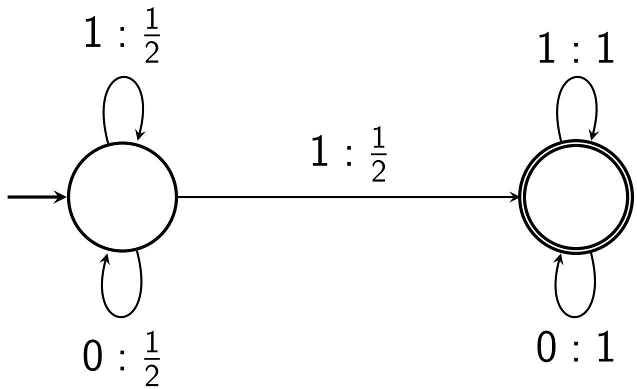

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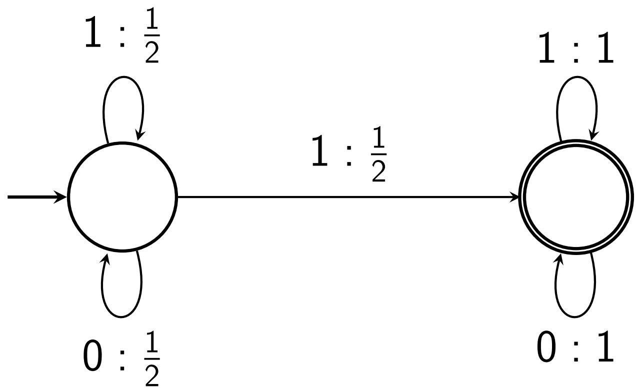

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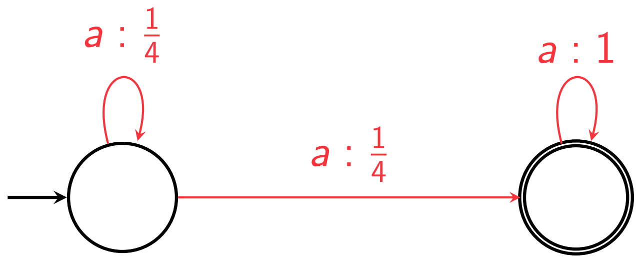

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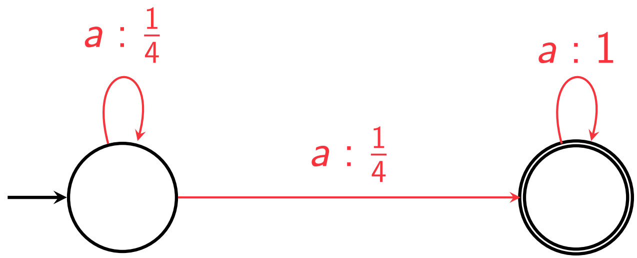

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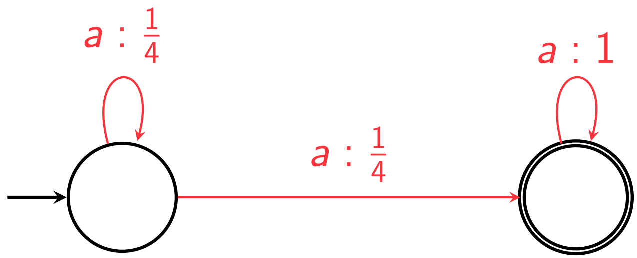

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- Then $\mathcal{A}$ defined as $\mathcal{A}_{\varphi_1} - \mathcal{A}_{\varphi_2}$ has the property that $\mathcal{A}(w) = 0$ iff the Post correspondence instance is valid

■