Lecture 9

Equivalence of weighted automata over fields is in PTIME

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Proof.

Let $\mathcal{C} = \mathcal{A} - \mathcal{B} = \{d, \Sigma, \{M_a\}_{a \in \Sigma}, I, F\}$

We want to know if $\mathcal{C}(w) = 0$ for all words w

Equivalence of weighted automata (2)

Recall that $\mathcal{C}(a_1 \ldots a_n) = I^\top M_{a_1} \ldots M_{a_n} F$

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We define the following sequence of $X_i \subseteq \mathbb{Q}^d$

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- $u \in X_{k+1} \iff u \in X_k$ and $M_a \cdot u \in X_k$ for all $a \in \Sigma$

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Claim

1. $X_0 \supseteq X_1 \supseteq X_2 \dots$
2. X_k is a vector space for every k

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- Let $X = \bigcap_k X_k$

Equivalence of weighted automata (3)

Lemma

1. If $X_{k+1} = X_k$ then $X_{k+2} = X_{k+1}$
2. $X = X_d = X_{d+1} = \dots$
3. For all k

$$F \in X_k \text{ iff } \mathcal{A}(w) = 0 \text{ for all } w \in \Sigma^{\leqslant k}$$

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  How to iteratively compute the basis?

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- It remains to prove the Lemma

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Since $u \in X_{k+1}$ then $M_a \cdot u \in X_k$ for all a

But since $X_{k+1} = X_k$ we get $M_a \cdot u \in X_{k+1}$

Equivalence of weighted automata (6)

2. $X = X_d = X_{d+1} = \dots$

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This is trivial because

X_k are vector spaces

$dim(X_0) \leqslant d$

Previous property

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  So by induction assumption $I \cdot M_w \cdot M_a \cdot u = 0$

  Done because $M_w \cdot M_a = M_{w'}$ ■