Tutorials 3

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1 Exercise

Recall the three classes of sequences from Lecture 2:

1. Homogenous linear recurrence sequences.
2. Nonhomogenous linear recurrence sequences.
3. Sequences defined with a system of linear sequences.

We proved that over commutative rings these classes are equivalent and that for any semiring $(1) \subseteq (2) \subseteq (3)$. Show that over the semiring $(\mathbb{N}, +, \cdot, 0, 1)$ the inclusions are strict: $(1) \subsetneq (2) \subsetneq (3)$. That is, find a sequence in (3) that is not in (2) and a sequence in (2) that is not in (1).

2 Exercise

Prove that for any integers $n \geqslant 0$ and $k \geqslant 1$

$$\sum_{i=0}^{k+1} \binom{k+1}{i} (-1)^i \cdot (n+k+1-i)^k = 0.$$

Hint. It has something to do with Lecture 2.

3 Exercise

We denote sequences over a semiring $\mathbb{S}(\oplus, \odot, \mathbb{0}, \mathbb{1})$ by $\langle u_n \rangle_{n \in \mathbb{N}} = u_0, u_1, \ldots$ where $u_n \in \mathbb{S}$.

Consider the class $\mathcal{C}$ of sequences definable by systems of linear sequences over some semiring $\mathbb{S}(\oplus, \odot, \mathbb{0}, \mathbb{1})$. Prove that if $\langle u_n \rangle_{n \in \mathbb{N}}, \langle v_n \rangle_{n \in \mathbb{N}} \in \mathcal{C}$ then their pointwise product $\langle u_n \odot v_n \rangle_{n \in \mathbb{N}}$ and pointwise sum $\langle u_n \oplus v_n \rangle_{n \in \mathbb{N}}$ are also in $\mathcal{C}$.