Polynomial-Space Completeness of Reachability for Succinct Branching VASS in Dimension One

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Warsaw

BVASS

Recall VASS

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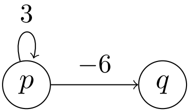

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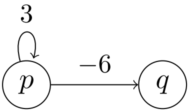

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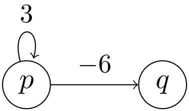

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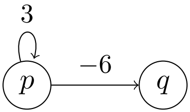

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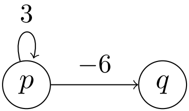

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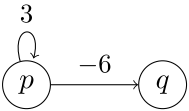

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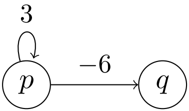

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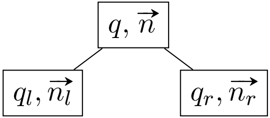

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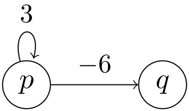

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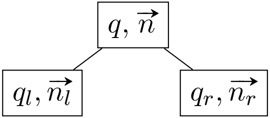

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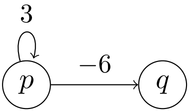

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unary/binary

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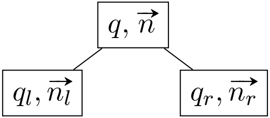

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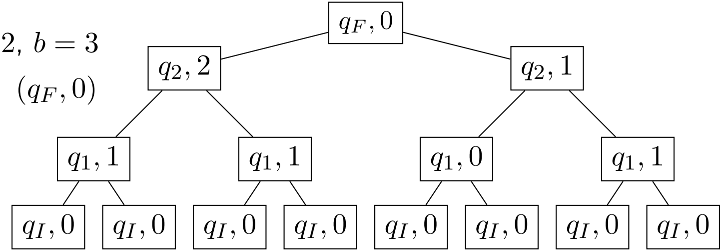

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Input: BVASS $\mathcal{B}$, configuration $(q, \vec{n})$

Problem: reachability of $(q, \vec{n})$

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Other problems:

Coverability, boundedness – 2EXPTIME-complete [Demri et al., 2013]

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Reachability for $d = 1$

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NP-hard (binary: 1-VASS → 1-BVASS)

in EXPTIME (1-BVASS: unary → binary)

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Connections with:

- Timed pushdown systems [Clemente et al., 2017]

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Core of the paper

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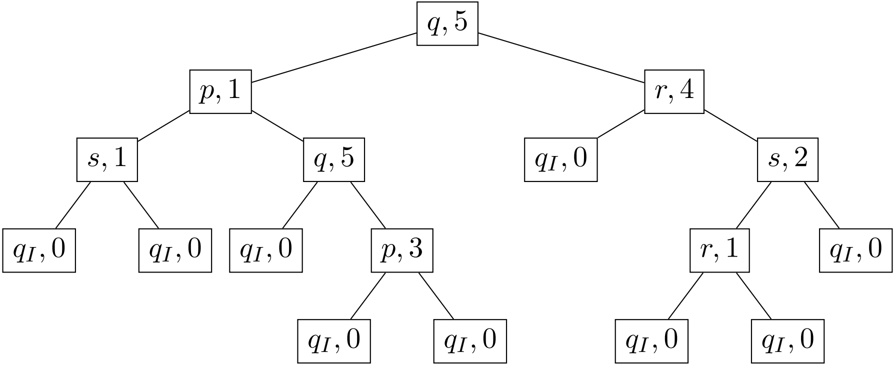

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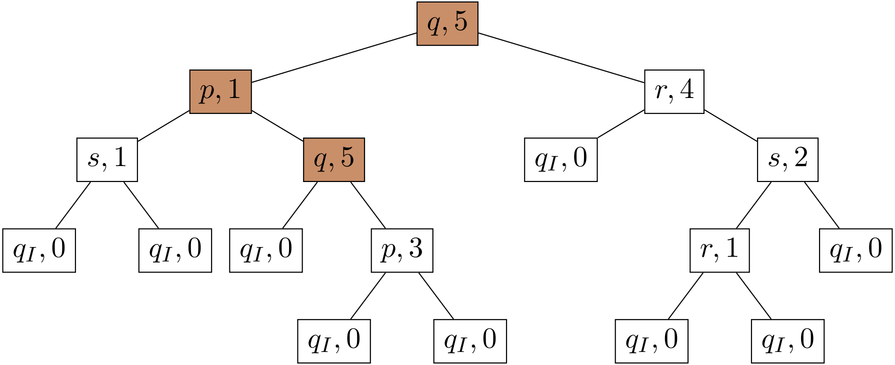

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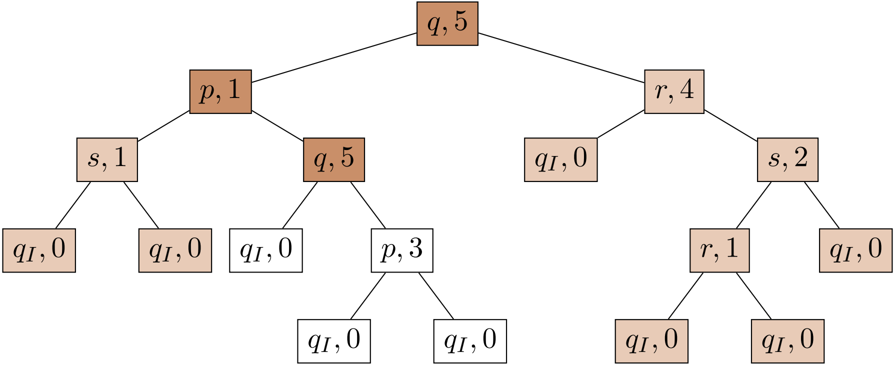

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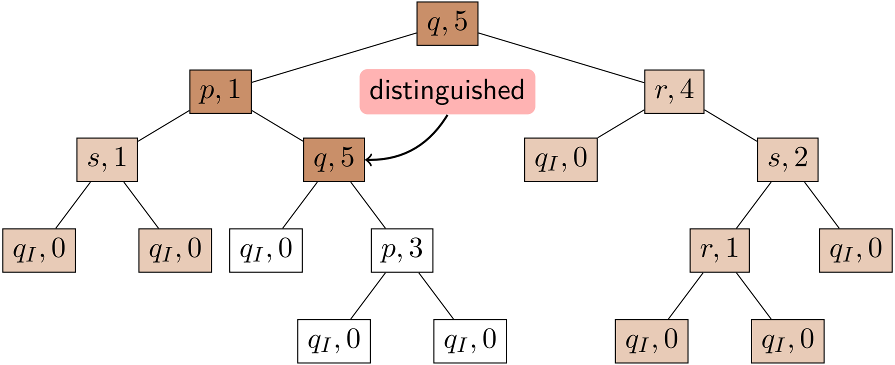

Cycle: run with a distinguished leaf

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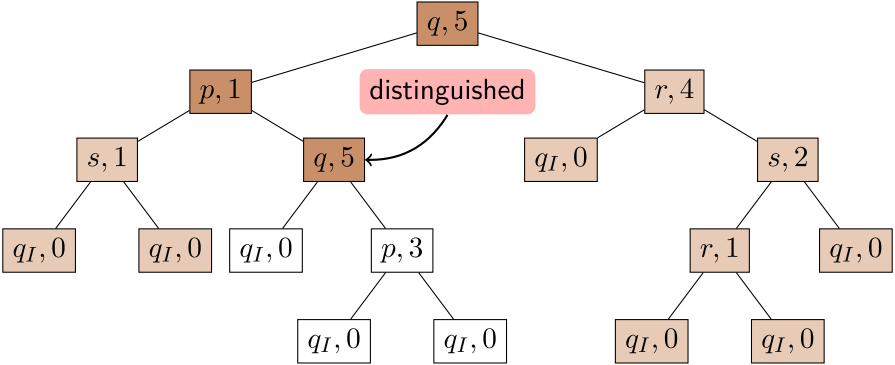

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Zero cycle

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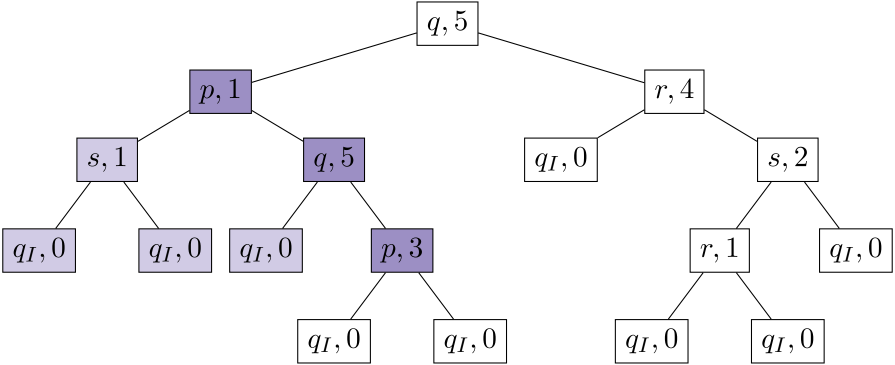

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Decreasing cycle (value -2)

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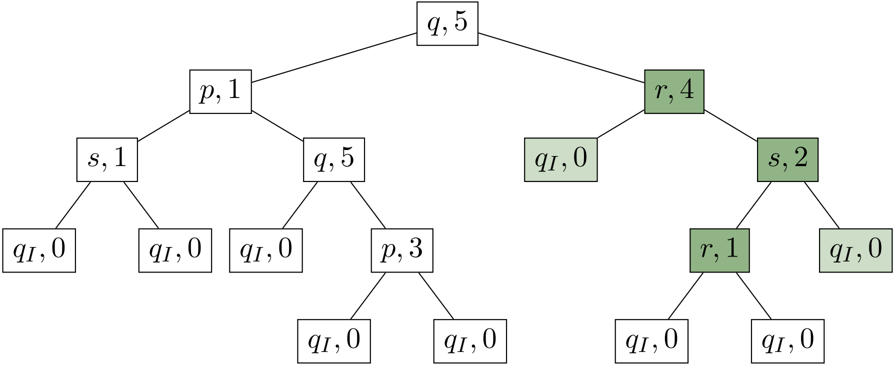

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Increasing cycle (value 3)

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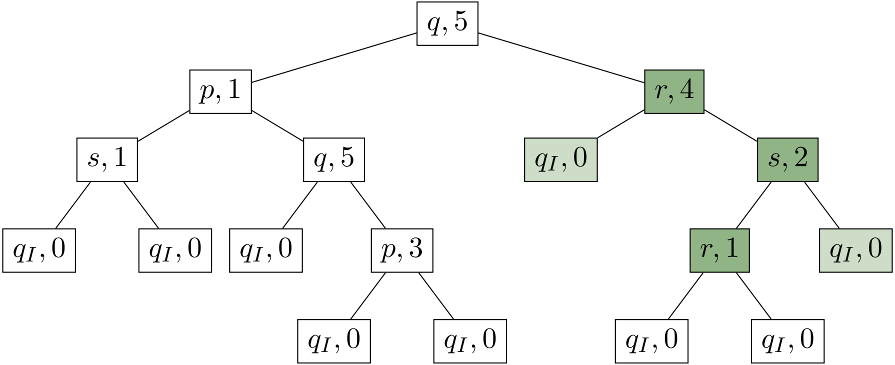

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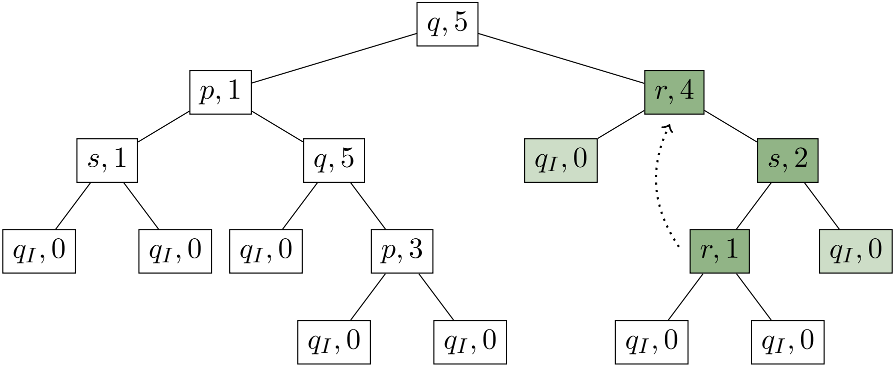

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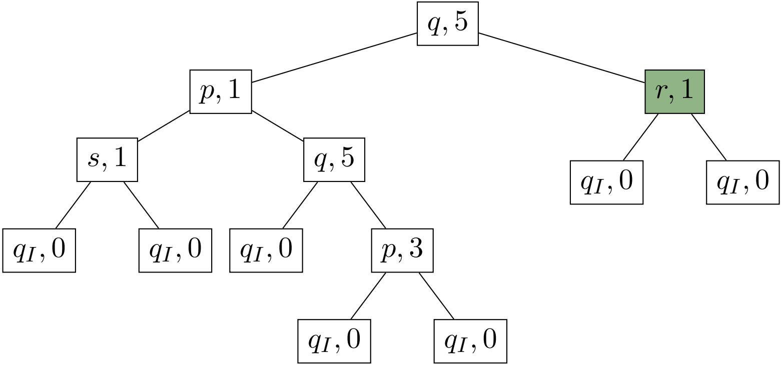

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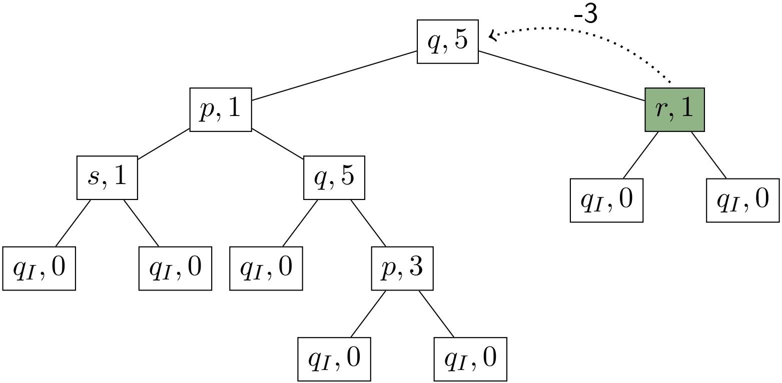

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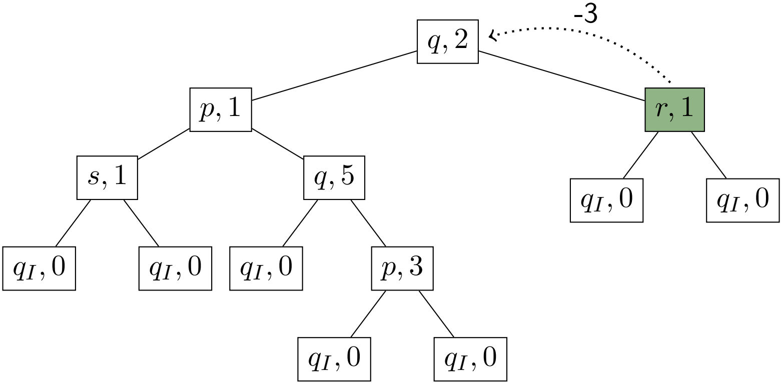

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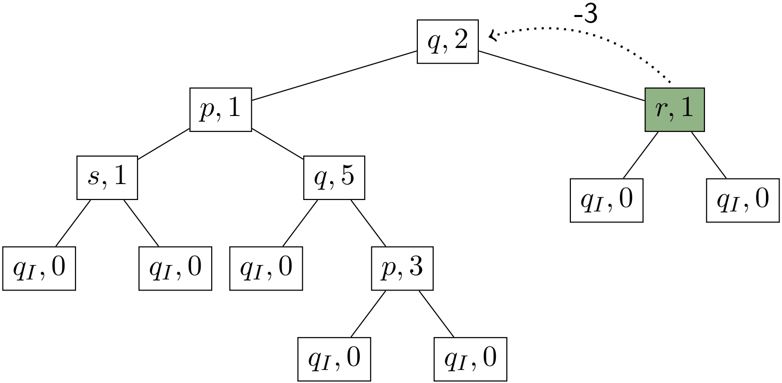

Cycle: run with a distinguished leaf

How to remove a cycle? Removing zero and decreasing cycles is safe

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Is (q, x) reachable, for some $x \geq n$

$x \geq n$ and $x \equiv n \mod d$

Lemma (small witness for coverability)

If (q, n) is d-coverable then there is a small computation.

Proof idea for coverability:

- Remove decreasing cycles and zero cycles
- Two cases:

Coverability modulo d

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Reachability

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Witness representing computations

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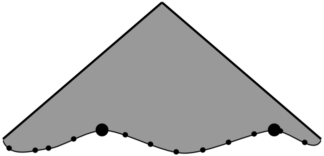
Witness representing computations

Partial run



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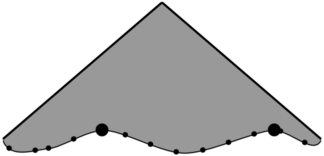
- proper leaves $(q_I, 0)$ $\bullet$



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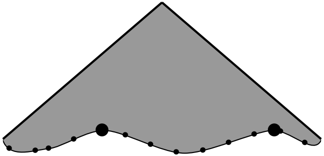

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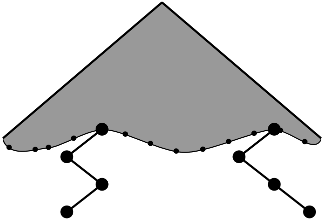
- for every ●



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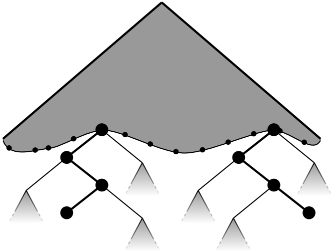

Cycles are implicit

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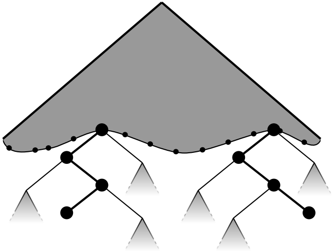

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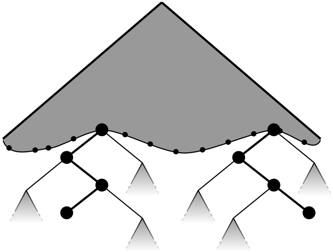
  (on bottom full runs)



How to get the computation tree?

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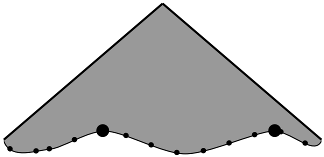

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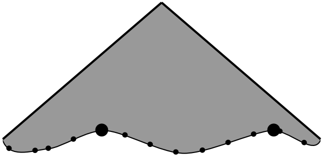

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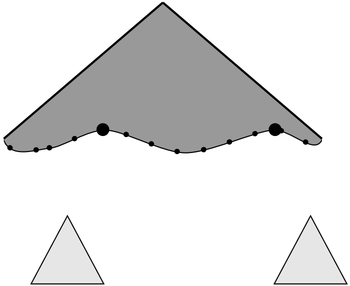
Build d_i-coverability runs



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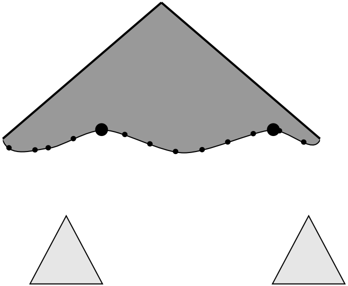

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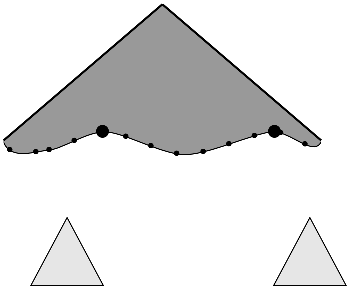

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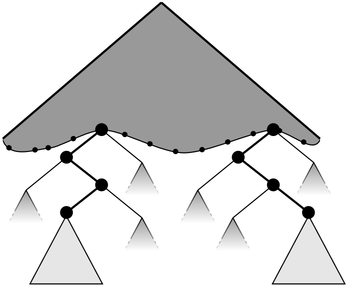

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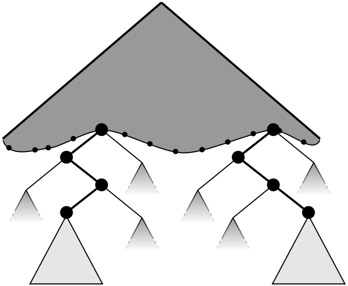

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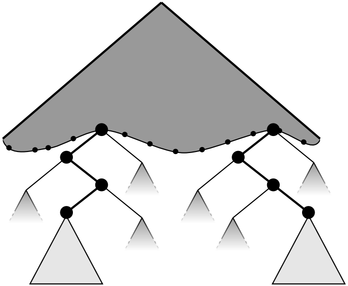

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Size: $\mathcal{O}(M^d)$, M – max partial run size, d – depth

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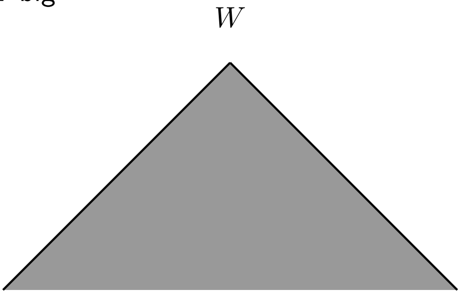

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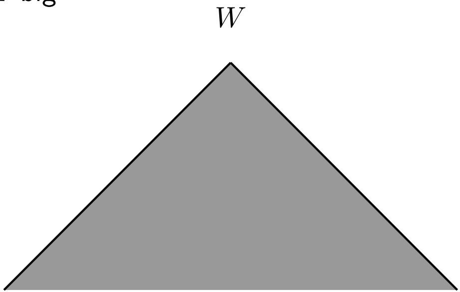

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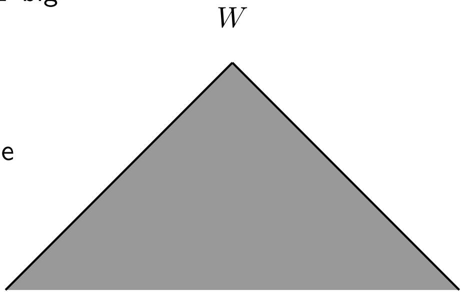

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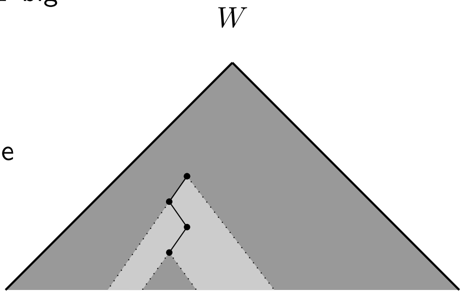

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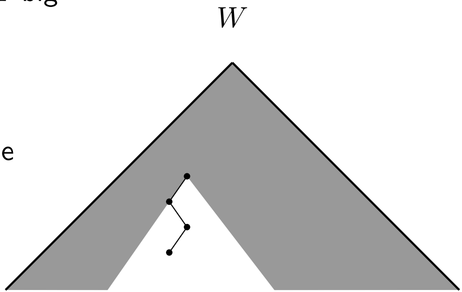

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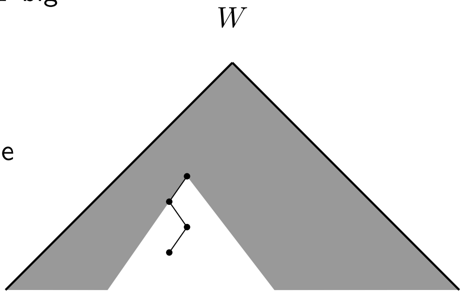

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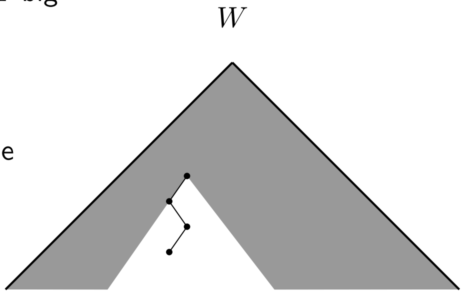
if $d > |Q|$ then collapse



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PSPACE-hardness

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Example $N = 4$, tape 1001

$$\boxed{(\underline{1}000)(\underline{0}000)(\underline{0}000)(\underline{1}000)}$$

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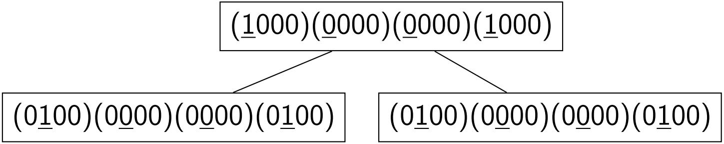

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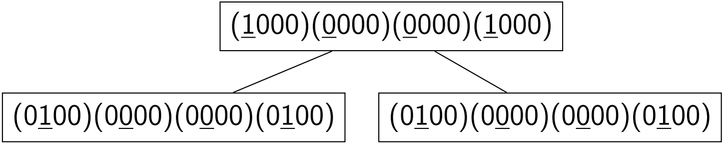

Branching with equal values

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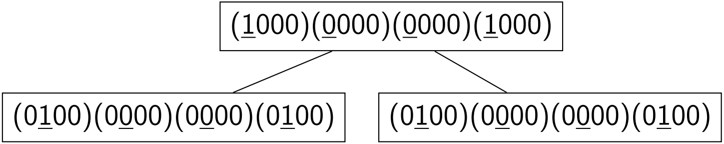

Branching with equal values

Possible for height N

Conclusions

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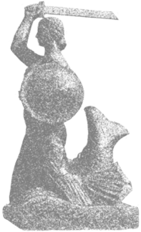

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