Reachability for Bounded Branching VASS

Filip Mazowiecki[1] and Michał Pilipczuk[2]

[1]University of Bordeaux

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CONCUR 2019

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Vector addition systems with states (d-VASS)

(Q, Δ), where $\Delta \subseteq Q \times \mathbb{Z}^d \times Q$

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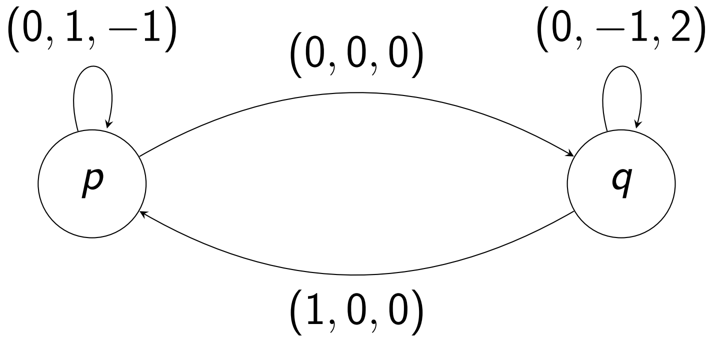
Example: $d = 3$, $Q = \{p, q\}$



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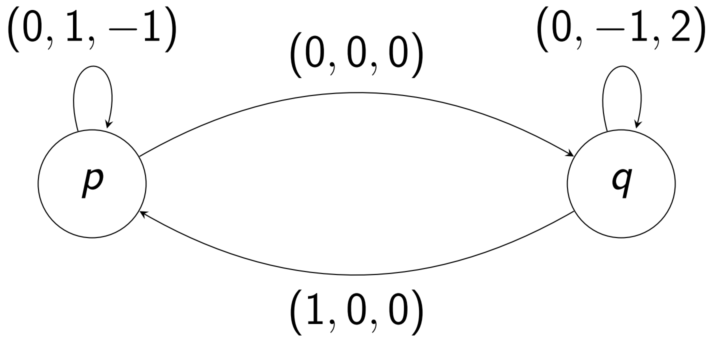

Configurations $p(\boldsymbol{v}) = (p, \boldsymbol{v}) \in Q \times \mathbb{N}^d$

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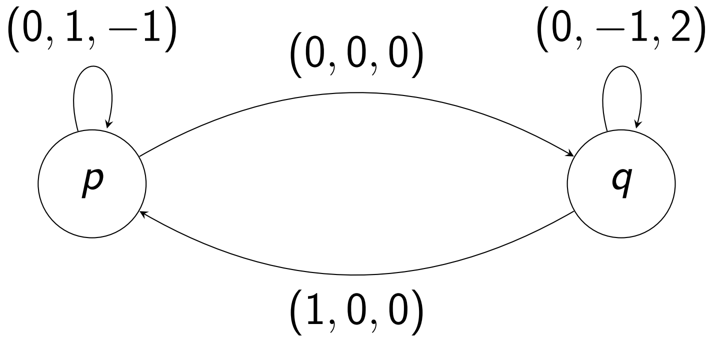

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Example run:

$p(0, 0, 1) \to p(0, 1, 0) \to q(0, 1, 0) \to q(0, 0, 2)$

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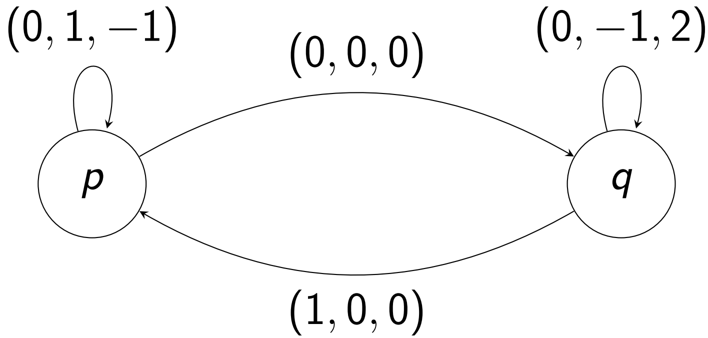

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Vector addition systems with states (d-VASS)

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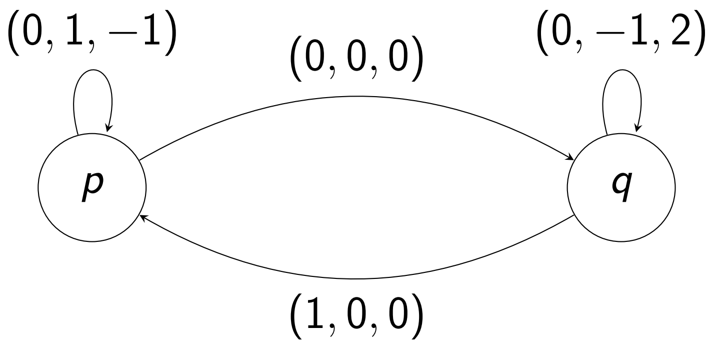

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(Q, Δ, B), where (Q, Δ) is a d-VASS

and $B \in \mathbb{N}$ (in binary)

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- Now we can also encode $c_i \leq n$ tests

  First $(0, \ldots, 0, B - n, 0, \ldots, 0)$ then $(0, \ldots, 0, n - B, 0, \ldots, 0)$

Decision problem

Reachability problem:

GIVEN: d-VASS (Q, Δ) and configurations $p(\boldsymbol{u}), q(\boldsymbol{v})$

(or d-BoVASS (Q, Δ, B))

DECIDE: whether $p(\boldsymbol{u}) \rightarrow^* q(\boldsymbol{v})$?

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- d-BOVASS: in PSPACE [Obvious]
  PSPACE-hard even for $d = 1$ [Fearnley and Jurdziński, 2013]

2-**BoVASS example**

Copying c_1 to c_2

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Copying c_1 to c_2

Let: $c_1 = n_1$, $c_2 = n_2$

$0 \leq n_1 \leq M$ and $B = M(M+2)$

Goal: $c_1 = c_2 = n_1$

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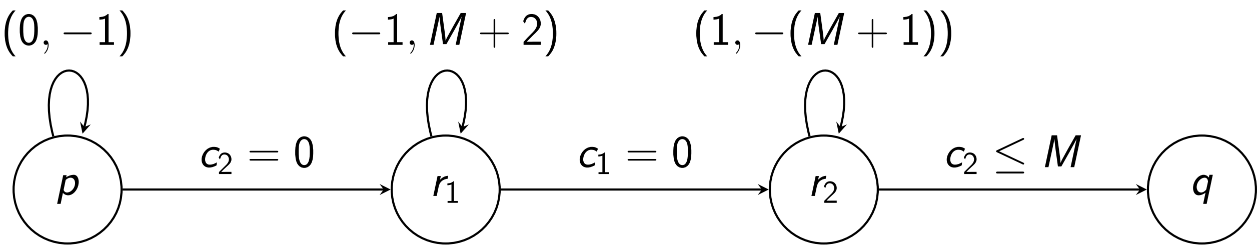

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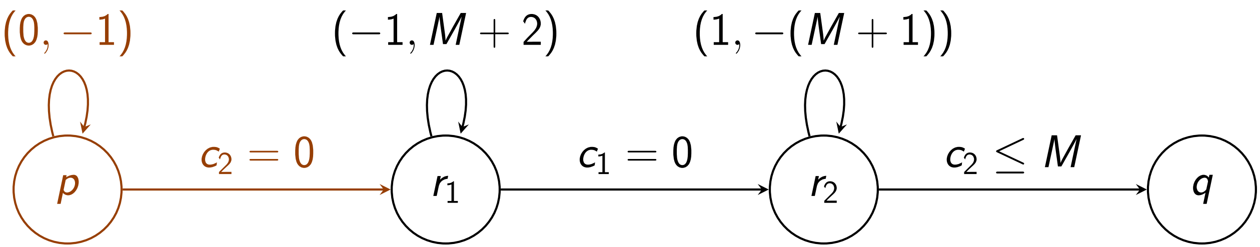

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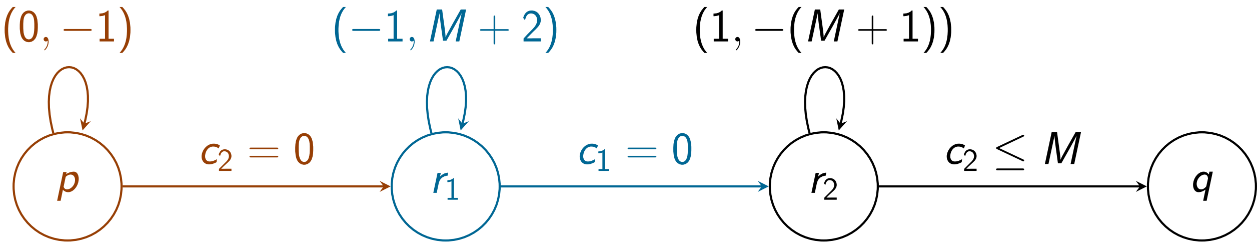

- $p(n_1, n_2) \to^* r_1(n_1, 0)$
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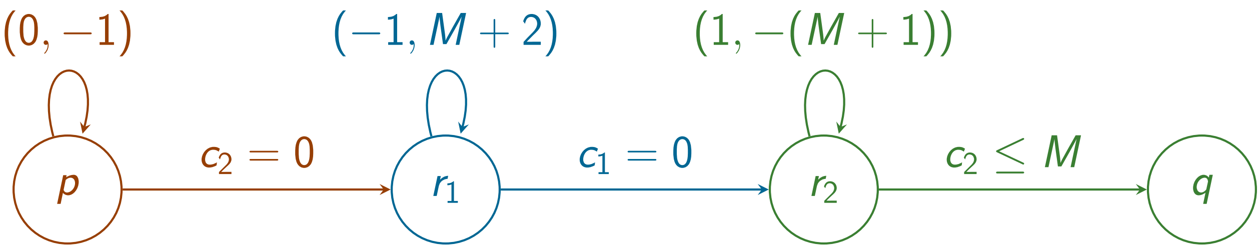

- $p(n_1, n_2) \to^* r_1(n_1, 0)$
- $r_1(n_1, 0) \to^* r_2(0, (M+2)n_1)$
- $r_2(0, (M+2)n_1) \to^* q(n_1, n_1)$

Branching d-VASS (d-BrVASS)

$(Q, \Delta_1, \Delta_2, q_0)$

where $\Delta_1 \subseteq Q \times \mathbb{Z} \times Q$, and $\Delta_2 \subseteq Q^3$

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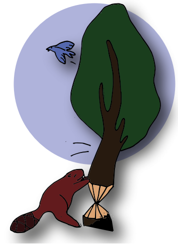

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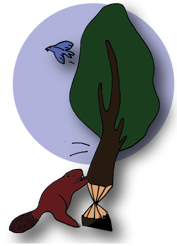

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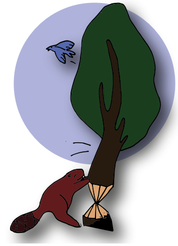

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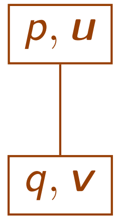
  - ▷ with one child: $(p, z, q) \in \Delta_1$, $\boldsymbol{u} + \boldsymbol{z} = \boldsymbol{v}$



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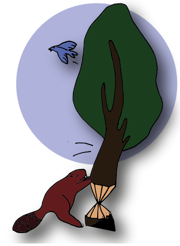

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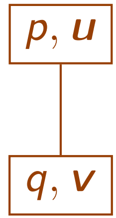
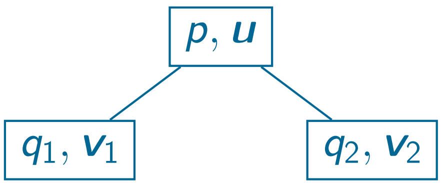


  - ▷ with two children: $(p, q_1, q_2) \in \Delta_2$, $\boldsymbol{u} = \boldsymbol{v}_1 + \boldsymbol{v}_2$



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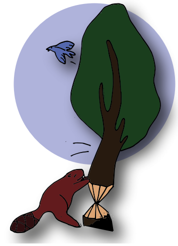

$(Q, \Delta_1, \Delta_2, q_0)$

where $\Delta_1 \subseteq Q \times \mathbb{Z} \times Q$, and $\Delta_2 \subseteq Q^3$

Configurations: $Q \times \mathbb{N}^d$ (the same)

If $\Delta_2 = \emptyset$
then it's a d-VASS

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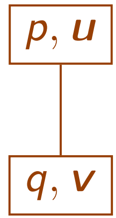

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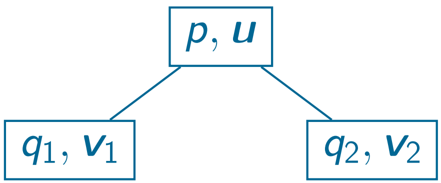

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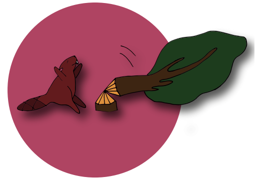

Fix n, b $\qquad (n = 3,\ b = 3)$

- states $Q = \{q_1 \dots q_n\} \cup \{q_0, q_F\}$
- three types of transitions:
  - $(q_1, 0, q_0), (q_1, -1, q_0) \in \Delta_1$ (initialize)

$$\begin{array}{cccc} \boxed{q_1, 1} & \boxed{q_1, 1} & \boxed{q_1, 0} & \boxed{q_1, 1} \\ | & | & | & | \\ \boxed{q_0, 0} & \boxed{q_0, 0} & \boxed{q_0, 0} & \boxed{q_0, 0} \end{array}$$

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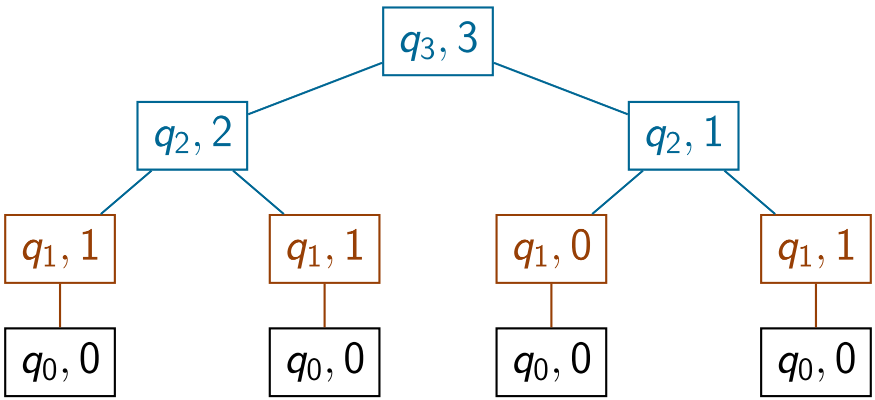

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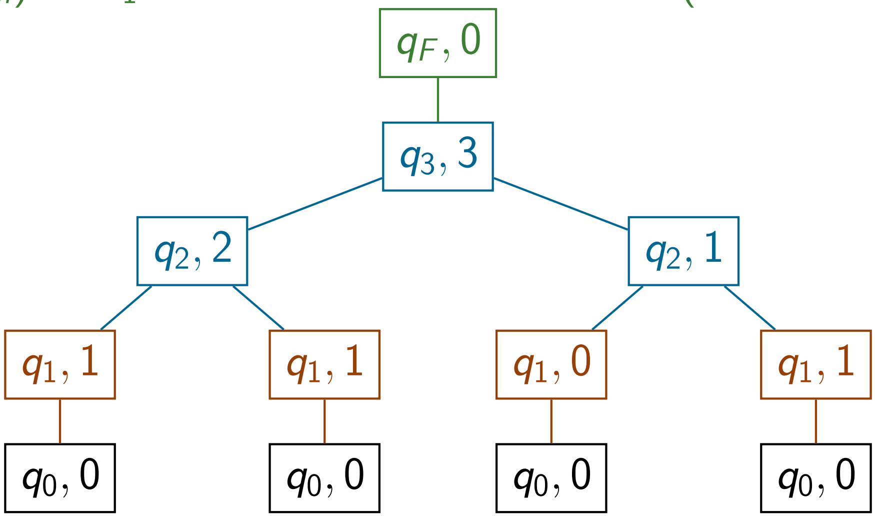
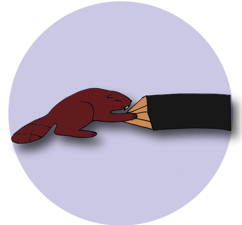
  - $(q_F, b, q_n) \in \Delta_1$ (check value)



Decision problem for d-BrVASS



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GIVEN: d-BrVASS $(Q, \Delta_1, \Delta_2, q_0)$ and a configuration $p(\boldsymbol{u})$

DECIDE: is there a computation tree with $p(\boldsymbol{u})$ in the root?

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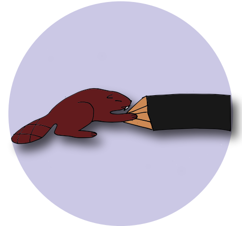

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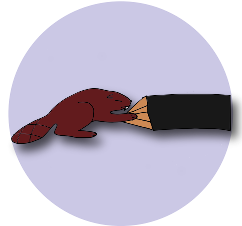

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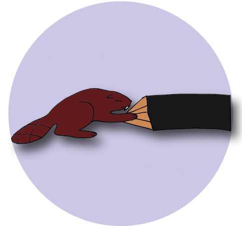

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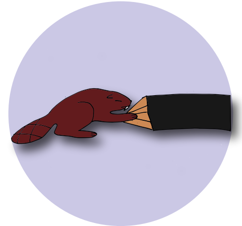

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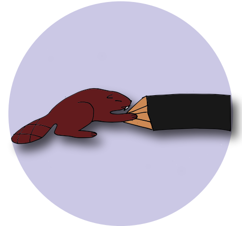

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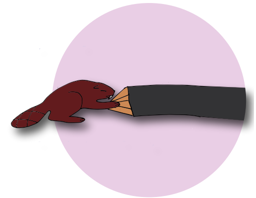
in EXPTIME [Obvious]

Results for reachability for d-BoBrVASS



For $d = 1$

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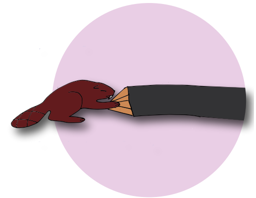

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	unary	binary	with bound
VASS	NL-complete	NP-complete	PSPACE-complete

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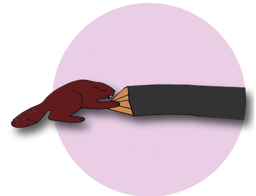

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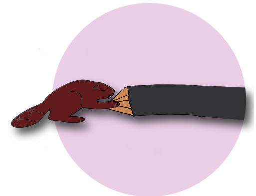

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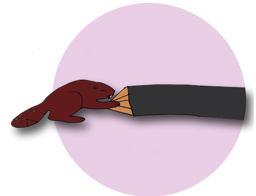

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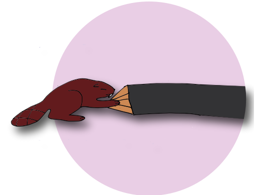

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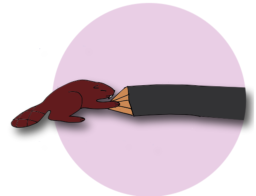

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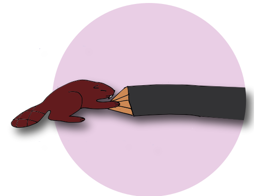

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- Reachability for 1-BoBrVASS remains open

The 2-BoBrVASS we work with

$(Q, \Delta_1, \Delta_2, q_0)$

c_1 – the real counter, c_2 – buffer (usually $c_2 = 0$)

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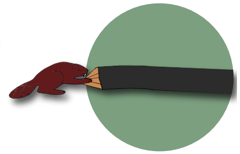

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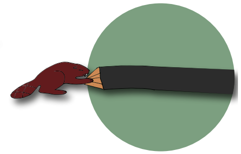

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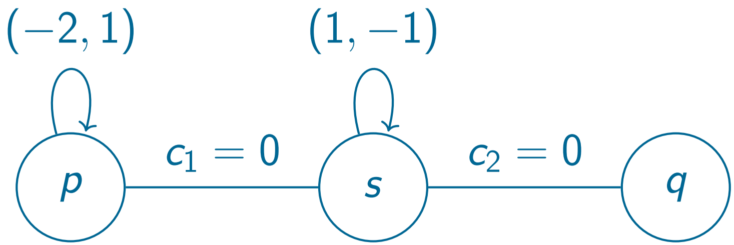

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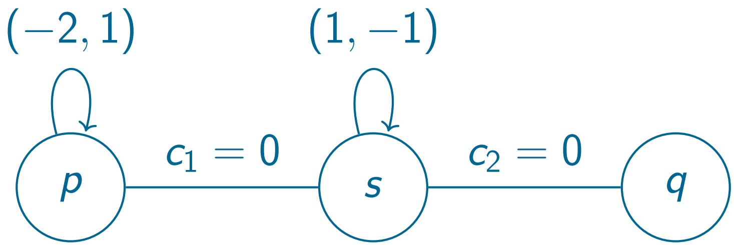

Notation: 1-$\textsc{BoBrVASS}^{\div 2}$

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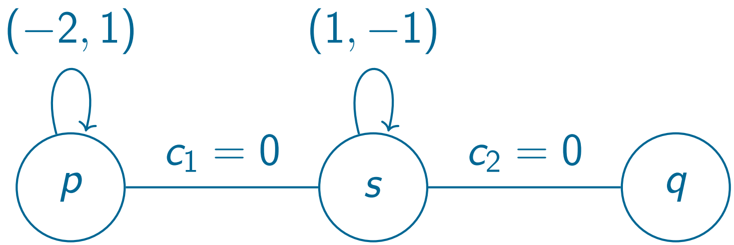

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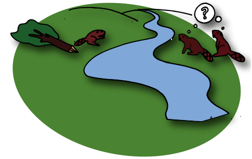
Remark

$$\text{1-BoBrVASS}^{\div 2} = \text{1-BoBrVASS}^{\times 2} = \text{1-BoBrVASS}^{\times 2, \div 2}$$

Two gadgets for 1-BoBrVASS$^{\div 2}$



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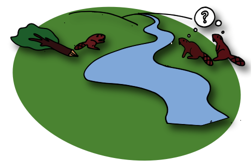

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$$\begin{tikzcd}
\boxed{p} \arrow[loop above, "M+M^2+M^3"] \arrow[r, "c_1 \leq M^4"] & \boxed{r} \arrow[loop above, "-1-M^3"] \arrow[r, "c_1 \leq M^3 + M^2"] & \boxed{q_0} \arrow[r, "\div 2"] & \cdots \arrow[r, "\div 2"] & \boxed{q_n}
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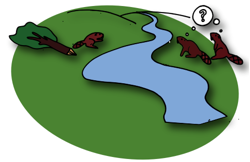

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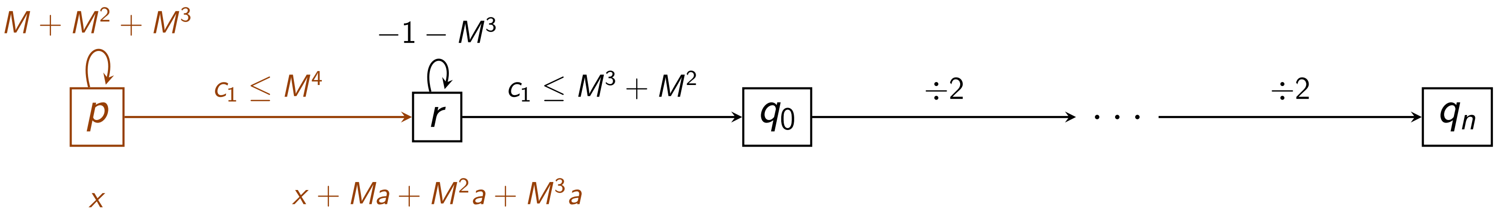

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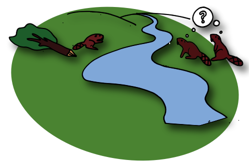

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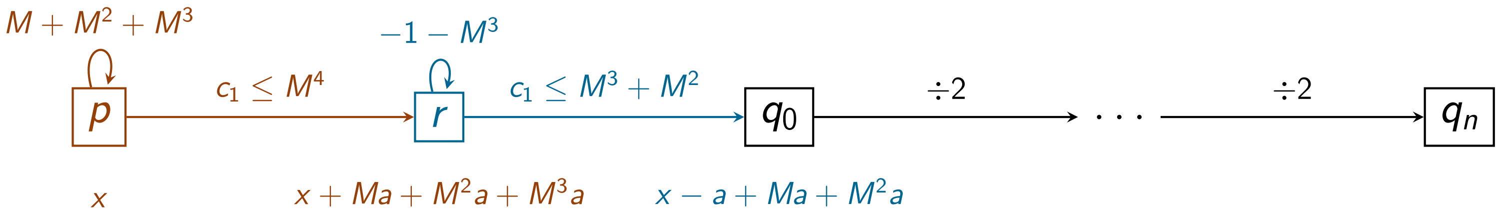

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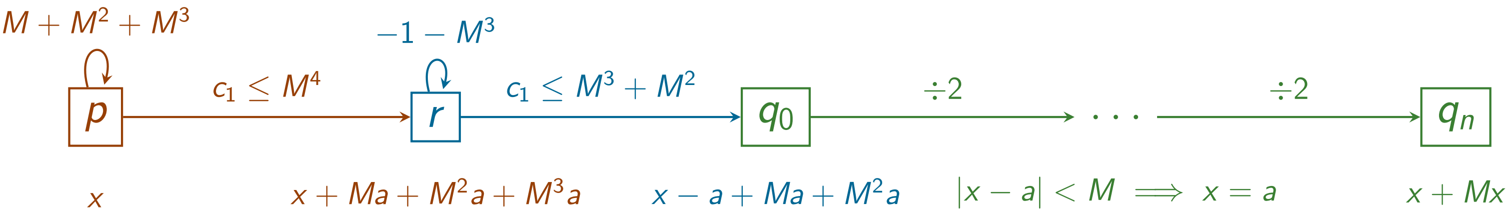

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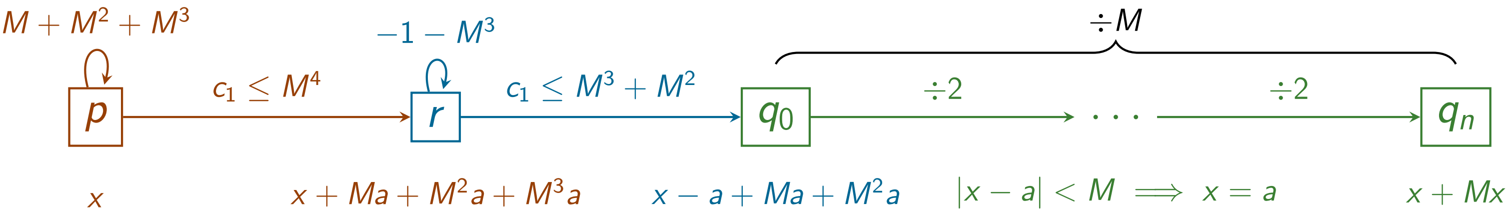

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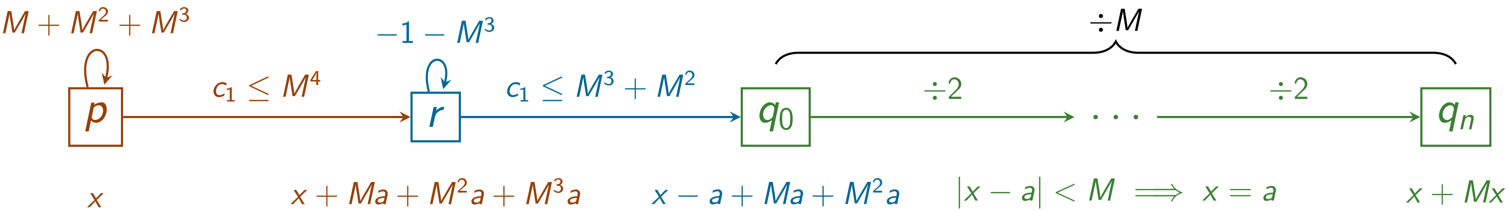

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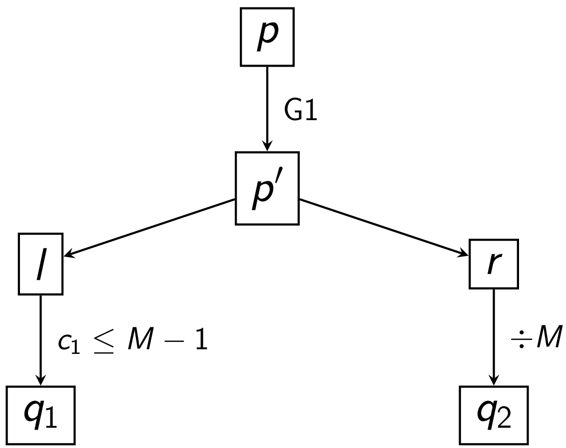

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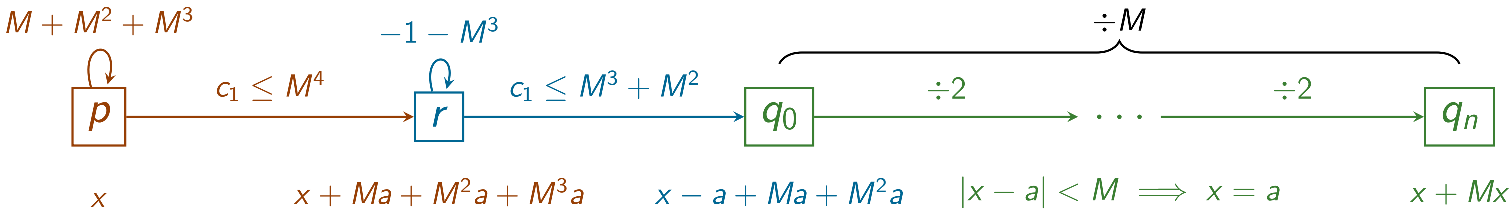

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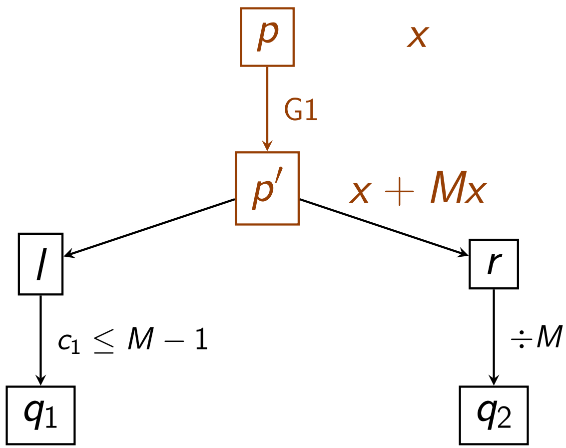

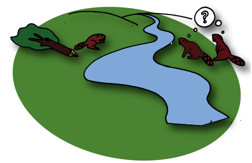

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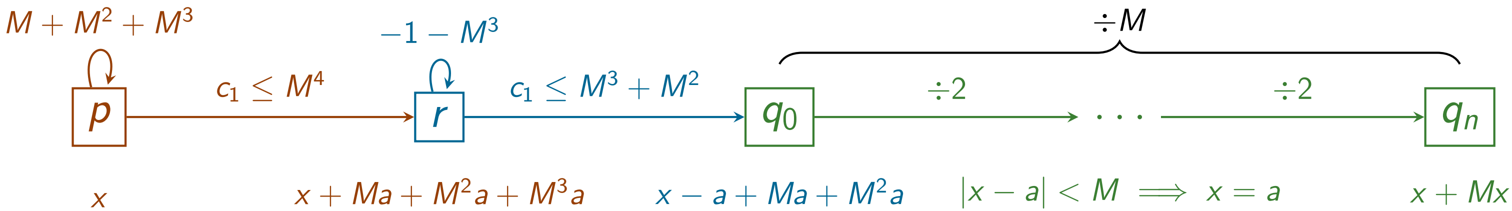

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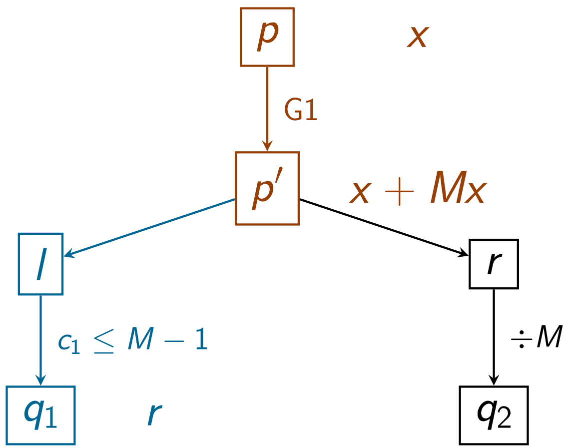

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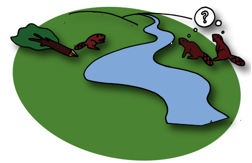

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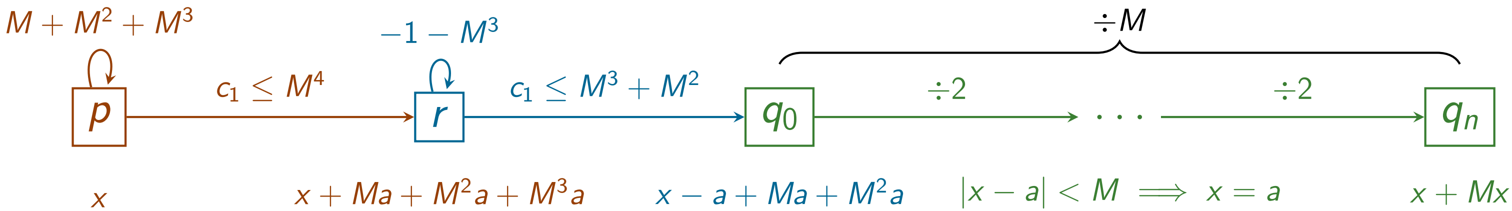

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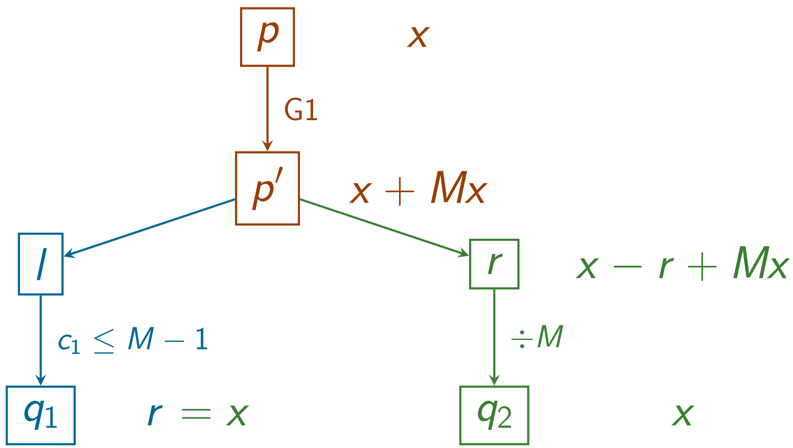

Countdown games

$(S, T), \quad S = S_\exists \uplus S_\forall$

$T \subseteq S \times \mathbb{N} \setminus \{0\} \times S$ (in binary)

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Lemma (Jurdziński et al., 2008)

GIVEN: (S, T) and $s(c)$ DETERMINE: does $\exists$ have a winning strategy

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Given (S, T) and $s(c)$ $\quad (c < M = 2^n)$

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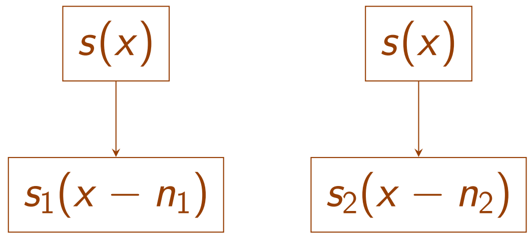

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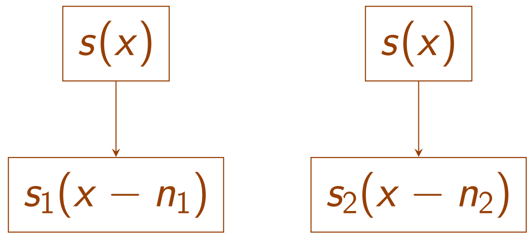

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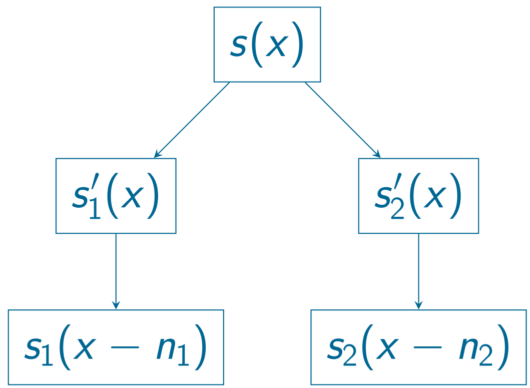

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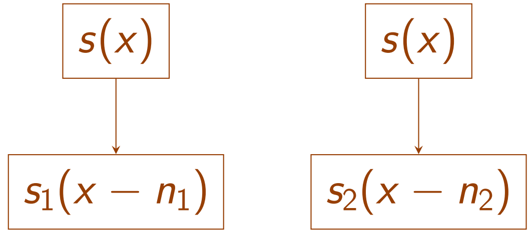

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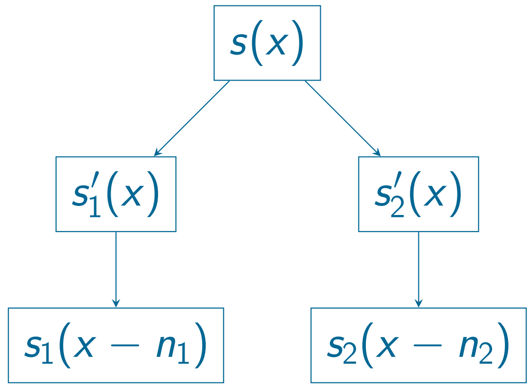

- For every $s \in S$

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Approaches for 1-BoBrVASS

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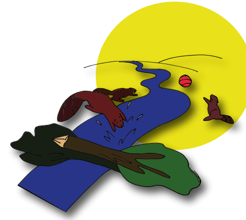
- Is it in PSPACE?

  It suffices to show that computation trees can be of exponential size

Approaches for 1-BoBrVASS



Can we simulate $\div 2$ or $\times 2$ in 1-BoBrVASS?

We prove that no

- Is it EXPTIME-hard?

  Then one should encode a full binary tree of exponential height

  A gadget like G2 would be useful

- Is it in PSPACE?

  It suffices to show that computation trees can be of exponential size

  That's how PSPACE upper bound for 1-BrVASS is proved

  Nothing from [Figueira et al., 2017] work for 1-BoBrVASS

Conclusion

- 2-BoBrVASS is EXPTIME-complete

  What about 1-BoBrVASS (In EXPTIME, PSPACE-hard)?

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- Some explanation: 'bobr' is 'beaver' in polish