Copyless Cost-Register Automata

Filip Mazowiecki

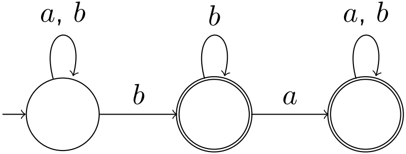
University of Warwick

Santiago 2016

Introduction

(mostly weighted automata)

Automata



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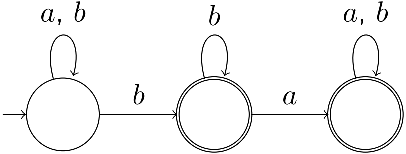
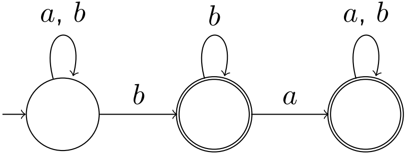


$f : \Sigma^* \to \{0, 1\}$

Automata



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Weighted automata

$f : \Sigma^* \to$ “some numbers”?

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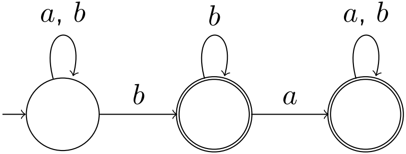

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$f : \Sigma^* \to$ "some numbers"? $\mathbb{N}$?

Commutative semirings

$\mathbb{S}(\oplus, \odot, \mathbb{0}, \mathbb{1})$ with some axioms $s \oplus \mathbb{0} = s,\ s \odot \mathbb{1} = s,\ \ldots$

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Examples:

- $\mathbb{S} = \mathbb{N}(+, \cdot, 0, 1)$

Nothing fancy: $\oplus = +,\ \odot = \cdot,\ \mathbb{0} = 0,\ \mathbb{1} = 1$

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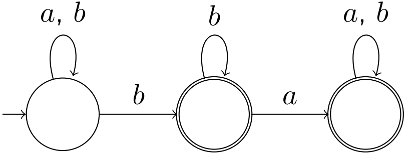
Weighted automata (WA)



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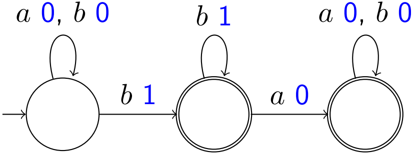

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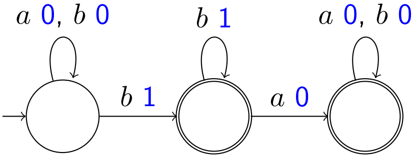

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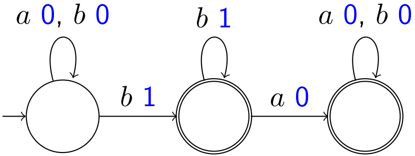

Consider $w = bbab$

b	b	a	b
1	1	0	0

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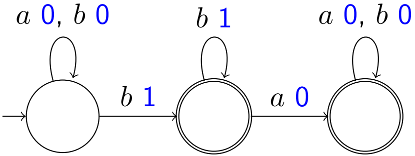

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b b a b

$1 + 1 + 0 + 0 = 2$

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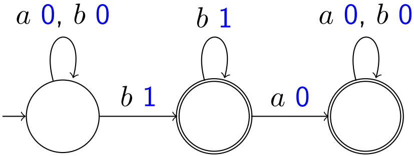

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1 +	1 +	0 +	0 = 2	0	1	0	0

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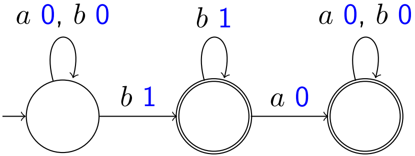

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$1 + 1 + 0 + 0 = 2$ $\qquad$ $0 + 1 + 0 + 0 = 1$

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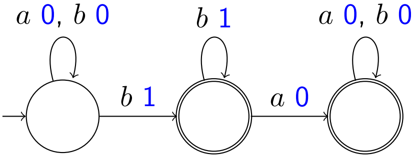

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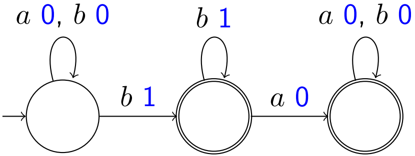

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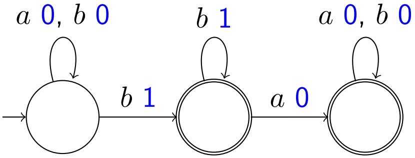

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b b a b: $1 + 1 + 0 + 0 = 2$

b b a b: $0 + 1 + 0 + 0 = 1$

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Output: $\max\{2, 1, 1\} = 2$

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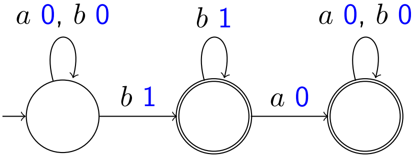

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In general: $\odot$ transitions, $\oplus$ accepting runs

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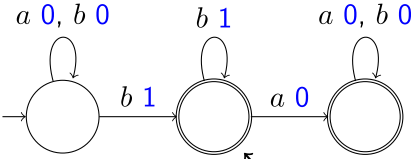

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WA subclasses

Bounding the number of accepting runs

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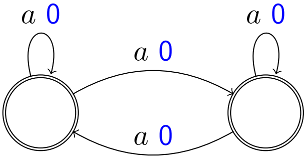

2^n accepting runs for a^n

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number of b's (linear)

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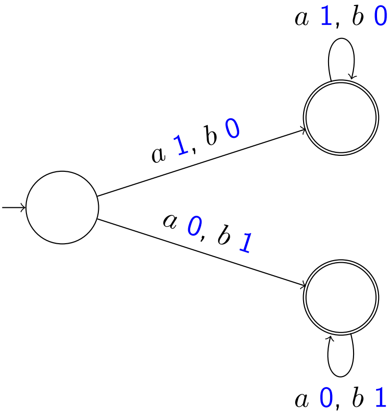

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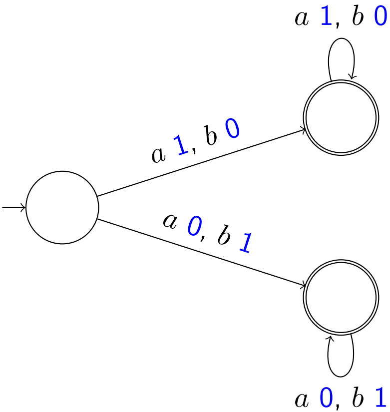

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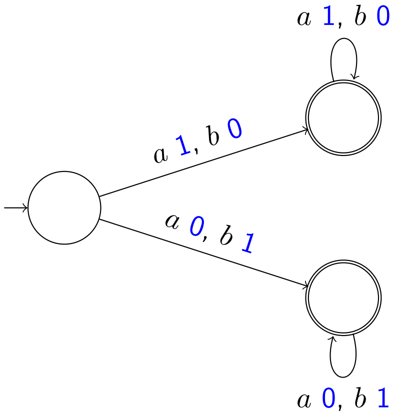

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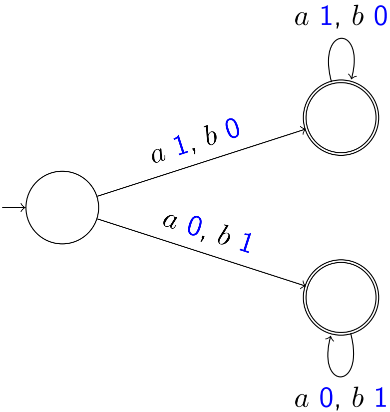
WA

$\supsetneq$

polynomially ambiguous WA



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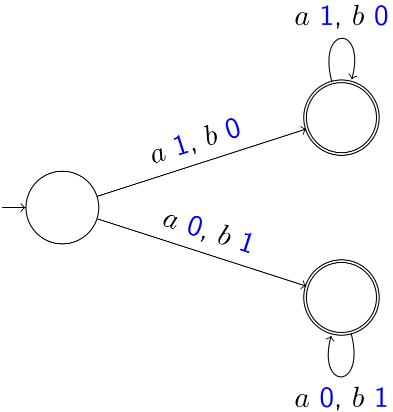

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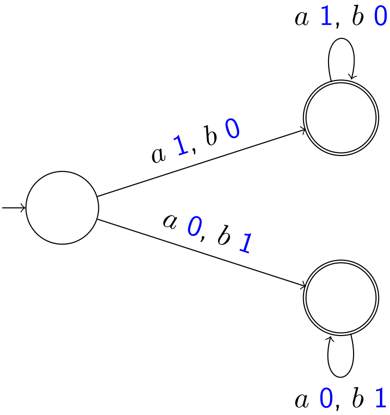

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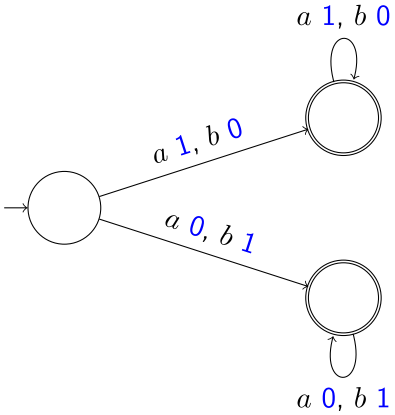
deterministic WA



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decidability border

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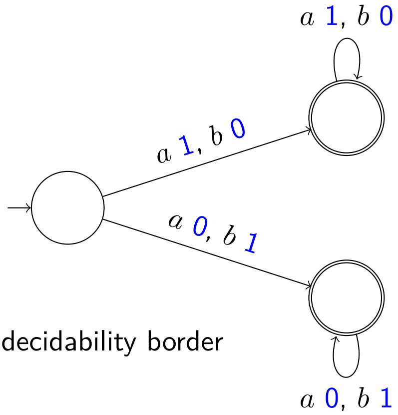

Logic characterizations

Monadic second-order logic (MSO) on words

$$\varphi \quad := \quad a(x) \;\mid\; x \leq y \;\mid\; x \in X \;\mid\; \neg\varphi \;\mid\; (\varphi \vee \varphi) \;\mid\; (\varphi \wedge \varphi) \;\mid\; Q$$

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MSO $=$ finite automata.

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Example in $\mathbb{N}_{-\infty}(\max, +)$

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Theorem

A fragment of WMSO is equivalent to WA.

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Intuition: $\bigoplus x.\ \bigodot y.\ \theta$

Cost register automata

(the model we work with)

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Deterministic automata with registers [Alur et al. 2013]

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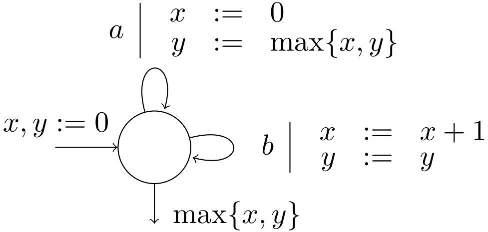

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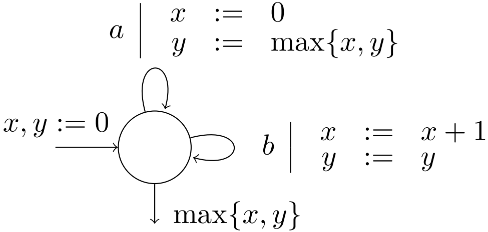

No constraints

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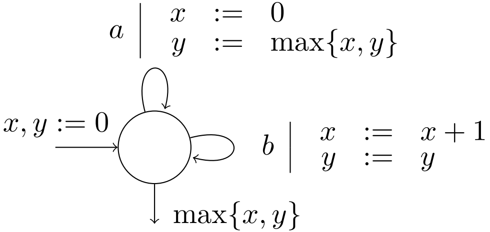

No constraints

No zero tests

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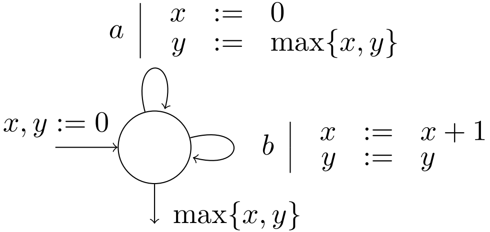


No constraints

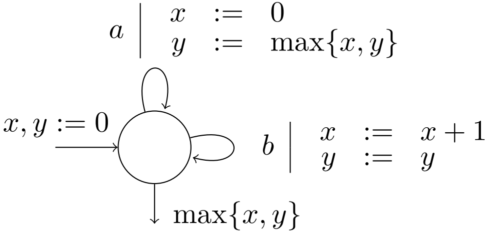
No zero tests

Initial $\longrightarrow$

$x = 0$
$y = 0$

Cost register automata (CRA)

Deterministic automata with registers [Alur et al. 2013]



No constraints

No zero tests

Initial	b
$x = 0$	$x = 1$
$y = 0$	$y = 0$

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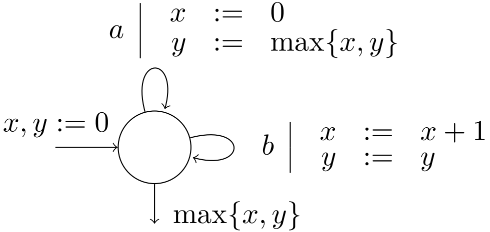

No constraints

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Initial		
	b	b
$x = 0$	$x = 1$	$x = 2$
$y = 0$	$y = 0$	$y = 0$

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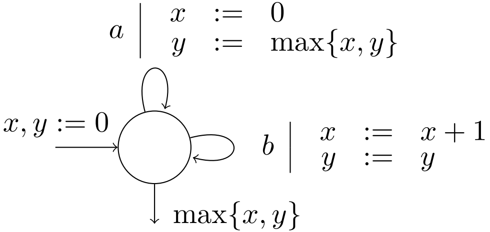

No constraints

No zero tests

Initial			
	b	b	a
$x = 0$	$x = 1$	$x = 2$	$x = 0$
$y = 0$	$y = 0$	$y = 0$	$y = 2$

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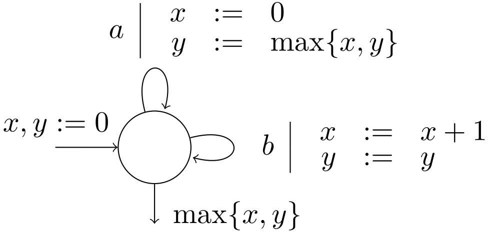

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Initial				
	b	b	a	b
$x = 0$	$x = 1$	$x = 2$	$x = 0$	$x = 1$
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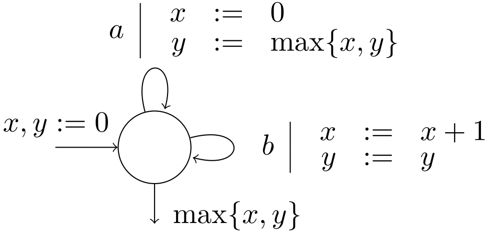

No constraints

No zero tests

Initial					Output
	b	b	a	b	
$x = 0$	$x = 1$	$x = 2$	$x = 0$	$x = 1$	2
$y = 0$	$y = 0$	$y = 0$	$y = 2$	$y = 2$	

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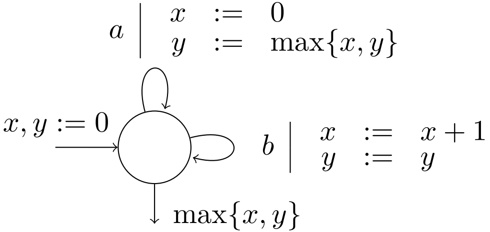

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x current block of b's

y previous maximal block of b's

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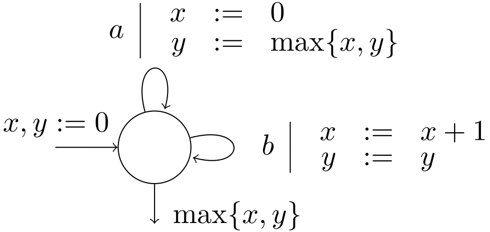

No constraints

No zero tests

Initial	b	b	a	b	Output
$x = 0$	$x = 1$	$x = 2$	$x = 0$	$x = 1$	2
$y = 0$	$y = 0$	$y = 0$	$y = 2$	$y = 2$	

x current block of b's

y previous maximal block of b's

"longest block of b's"

CRA over a unary alphabet

$\Sigma = \{a\}, \quad \mathbb{S} = \mathbb{N}(+, \cdot)$

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Words: $a^3, a^7, a^{18}, \ldots$

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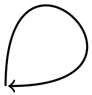

$$x := x \cdot y + z$$
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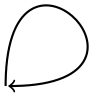

$$\begin{array}{l} x := x \cdot y + z \\ y := 3 \cdot z + 2 \\ z := x + 2 \cdot y \end{array} \quad \rightsquigarrow \quad \begin{cases} x(n+1) = x(n) \cdot y(n) + z(n) \\ y(n+1) = 3 \cdot z(n) + 2 \\ z(n+1) = x(n) + 2 \cdot y(n) \end{cases}$$

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$f(n) = F_n, \ g(n) = F_{n+1}$

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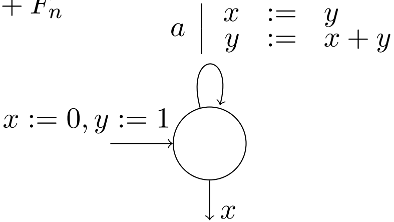

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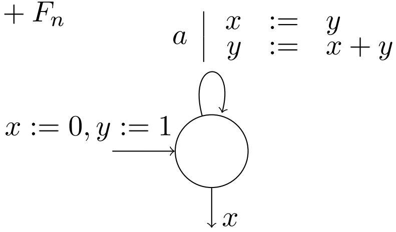

$[\![\mathcal{A}]\!](a^n) = F_n$

CRA are more than WA

Actually WA $\subsetneq$ CRA

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For example for $\mathbb{N}_{-\infty}(\max, +)$

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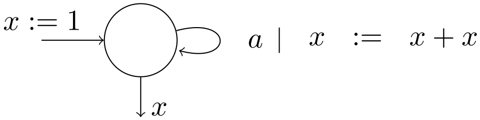

CRA are more than WA

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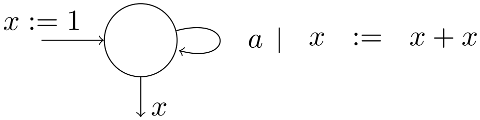

Output: $2^{|w|}$

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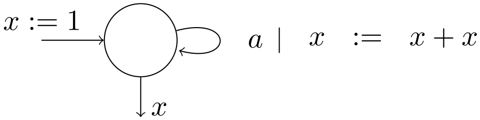

Output: $2^{|w|}$

For WA output $\in \mathcal{O}(|w|)$

Restrictions on CRA

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Restricted expressions

Operator $\odot$ only with constants

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Keep in mind!

In the semiring $\mathbb{N}_{-\infty}(\max, +)$:

$\odot = +$

$\oplus = \max$

Restrictions on CRA

Restricted expressions

Operator $\odot$ only with constants

$\max\{x, y\} + 3$
GOOD

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$\max\{x, y\} + 3$	$x + y$
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WA = CRA($\oplus$, $\odot s$) [Alur et al. 2013]

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Copyless restriction

Each register used only once

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Copyless restriction

Each register used only once

$x := x + y$
$y := 5$
GOOD

Restrictions on CRA

Restricted expressions

Operator $\odot$ only with constants

$\max\{x, y\} + 3$	$x + y$
GOOD	BAD

Keep in mind!
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Copyless restriction

Each register used only once

$x := x + y$	$x := \max\{x, y\}$
$y := 5$	$y := y$
GOOD	BAD

Copyless CRA

("Deterministic registers")

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How are they related to WA?

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Is there a logic characterization?

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In this talk

1. Function not recognizable by any Copyless CRA

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Copyless CRA

("Deterministic registers")

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In this talk

1. Function not recognizable by any Copyless CRA
2. Copyless CRA vs WA
3. Introduce BAC, a subclass of Copyless CRA

Copyless CRA

("Deterministic registers")

How are they related to WA?

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In this talk

1. Function not recognizable by any Copyless CRA
2. Copyless CRA vs WA
3. Introduce BAC, a subclass of Copyless CRA
4. Logic characterization for BAC

The counterexample

Set $\mathbb{N}_{-\infty}(\max, +)$, and $\oplus = \max$, $\odot = +$

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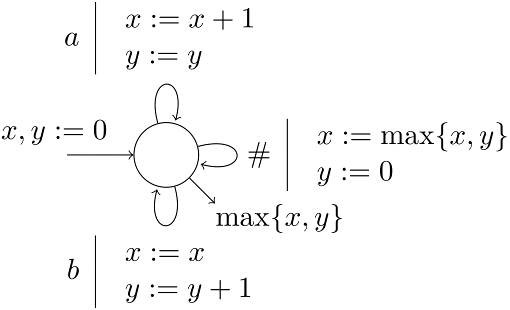

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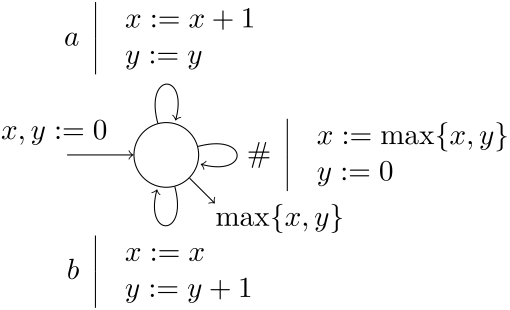

$ababab a\#aabbab^{10}a\#aab^7a\# \ldots$

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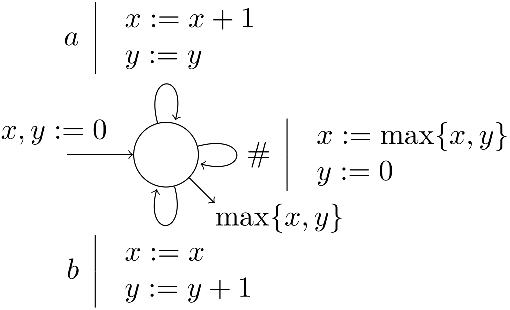
Set $\mathbb{N}_{-\infty}(\max, +)$, and $\oplus = \max$, $\odot = +$



$$\underbrace{ababab a}_{\text{block } 0}\#\underbrace{aabbab^{10}a}_{\text{block } 1}\#\underbrace{aab^{7}a}_{\text{block } 2}\#\ldots$$

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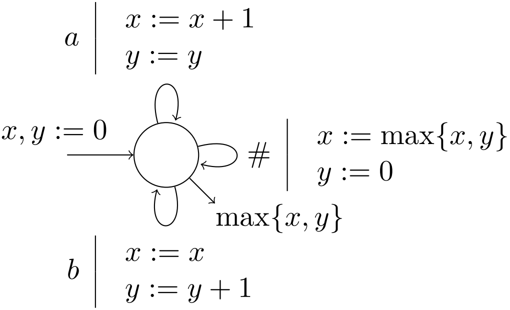

$$\overbrace{ababaaba}^{\text{block 0}}\#\overbrace{aabbab^{10}a}^{\text{block 1}}\#\overbrace{aab^{7}a}^{\text{block 2}}\#\ldots$$

y : "I just keep the number of b's in a block"

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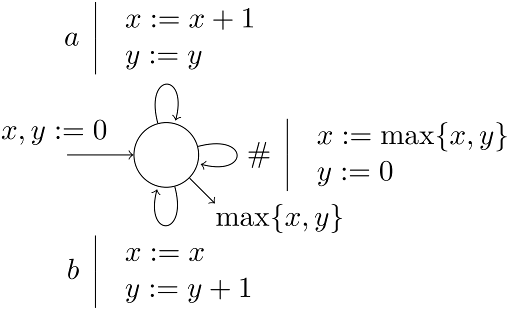

$$\overbrace{abababa}^{\text{block 0}}\#\overbrace{aabbab^{10}a}^{\text{block 1}}\#\overbrace{aab^7a}^{\text{block 2}}\#\ldots$$

y : "I just keep the number of b's in a block"

x : "I add 1 for every a and ..."

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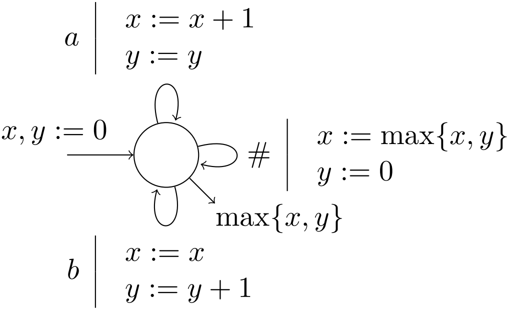
Set $\mathbb{N}_{-\infty}(\max, +)$, and $\oplus = \max$, $\odot = +$



$$\overbrace{ababaa}^{\text{block 0}}\#\overbrace{aabbab^{10}a}^{\text{block 1}}\#\overbrace{aab^{7}a}^{\text{block 2}}\#\ldots$$

$\uparrow$

$x = 0$

$y = 0$

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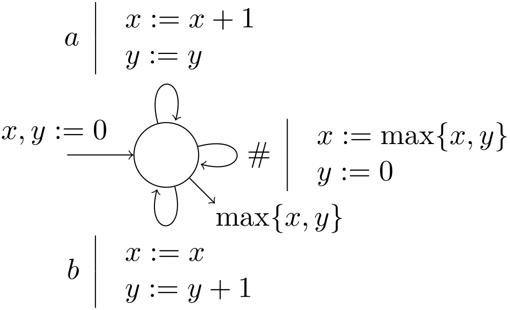
Set $\mathbb{N}_{-\infty}(\max, +)$, and $\oplus = \max$, $\odot = +$



$$\overbrace{ababa ba}^{\text{block 0}}\#\overbrace{aabbab^{10}}^{\text{block 1}}a\#\overbrace{aab^{7}a}^{\text{block 2}}\#\dots$$

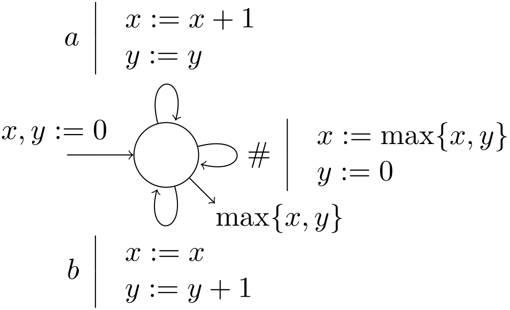
$\uparrow$

$x = 4$

$y = 3$

The counterexample

Set $\mathbb{N}_{-\infty}(\max, +)$, and $\oplus = \max$, $\odot = +$



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$\uparrow$

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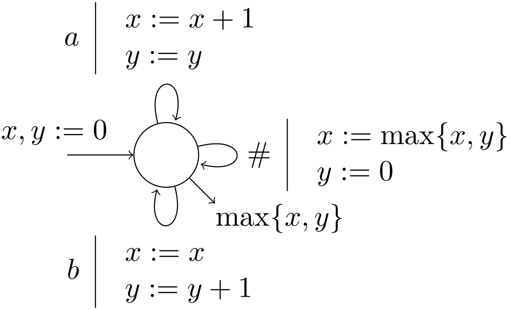


$$\overbrace{abababa}^{\text{block 0}}\#\overbrace{aabbab^{10}a}^{\text{block 1}}\#\overbrace{aab^{7}a}^{\text{block 2}}\#\dots$$

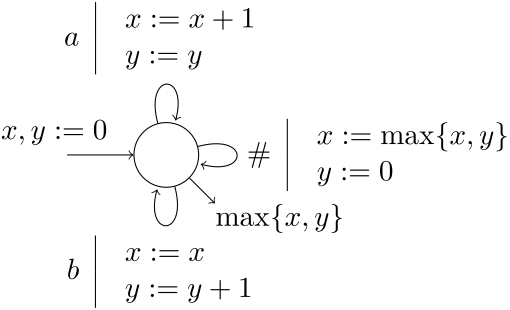
$\uparrow$

$x = 8$

$y = 12$

The counterexample

Set $\mathbb{N}_{-\infty}(\max, +)$, and $\oplus = \max$, $\odot = +$



$$\overbrace{ababab a}^{\text{block 0}}\#\overbrace{aabbab^{10}a}^{\text{block 1}}\#\overbrace{aab^7a}^{\text{block 2}}\#\dots$$

$\uparrow$

$x = 12$

$y = 0$

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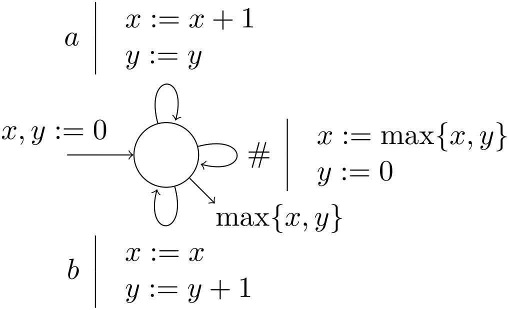
Set $\mathbb{N}_{-\infty}(\max, +)$, and $\oplus = \max$, $\odot = +$

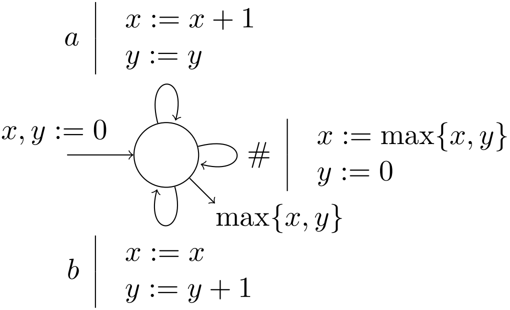


$$\overbrace{abababa}^{\text{block 0}}\#\overbrace{aabbab^{10}a}^{\text{block 1}}\#\overbrace{aab^{7}a}^{\text{block 2}}\#\ldots$$

$$x = 15$$
$$y = 7$$

The counterexample

Set $\mathbb{N}_{-\infty}(\max,+)$, and $\oplus = \max$, $\odot = +$



$$\overbrace{ababab a}^{\text{block 0}}\#\overbrace{aabbab^{10}a}^{\text{block 1}}\#\overbrace{aab^{7}a}^{\text{block 2}}\#\ldots$$

$\uparrow$

$x = 15$

$y = 0$

The counterexample (continued)

$$f(w) = \max_j \left\{ \, m_j + \sum_{i=j+1}^{k} n_i \right\}$$

m_i number of b's in block i
n_i number of a's in block i

The counterexample (continued)

$$f(w) = \max_j \left\{ \, m_j + \textstyle\sum_{i=j+1}^{k} n_i \right\}$$

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Can we do reverse?

$$f^R(w) = \max_j \left\{ \, \textstyle\sum_{i=0}^{j-1} n_i + m_j \right\}$$

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No.

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Copyless CRA are not closed under reverse

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- Observation: Copyless CRA $\subseteq$ WA

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Copyless CRA summary

Copyless CRA are not closed under reverse

- Observation: Copyless CRA $\subseteq$ WA

(Recall that WA $\subsetneq$ CRA)

WA are closed under reverse

$\implies$ copyless CRA $\subsetneq$ WA ☺

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- A logical characterization seems unlikely ☹

Bounded alternation copyless CRA (BAC)

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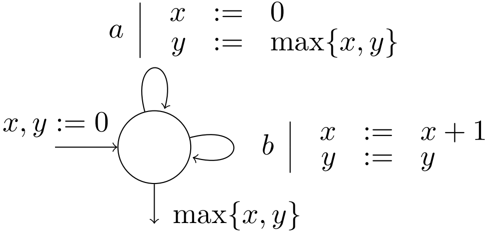

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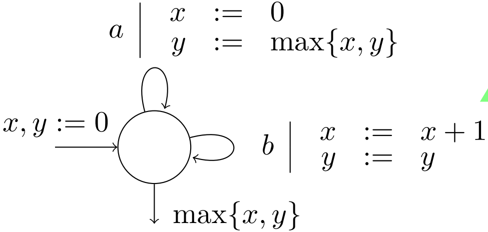

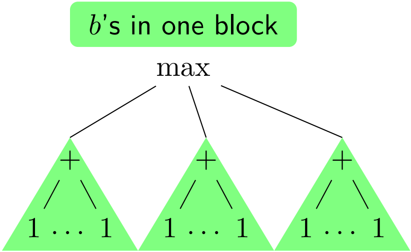

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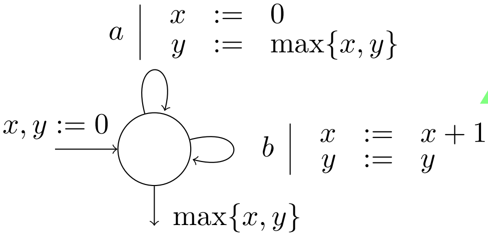
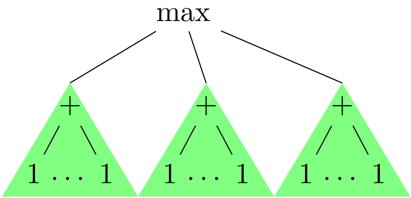


b's in one block



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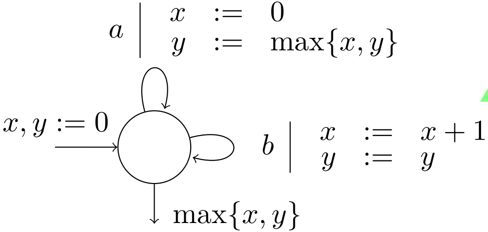

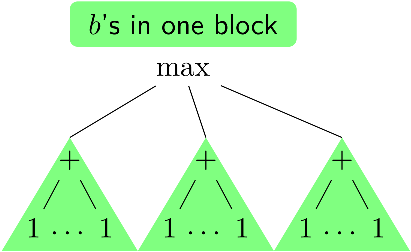

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BAC = copyless CRA + universally bounded alternation

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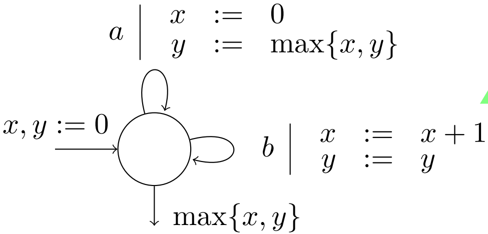

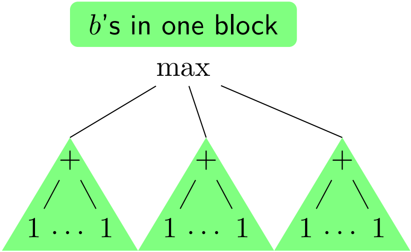

Alternation is 2

BAC = copyless CRA + universally bounded alternation

$\max\left\{ m_j + \sum_{i=j+1}^{k} n_i \right\}$ "simplest example" not in BAC

Closure properties

Maybe BAC is better?

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What extensions are interesting?

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- Nondeterminism [Cadilhac, Krebs, Limaye]
- Regular look-ahead [Alur et al.]

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Consider the function

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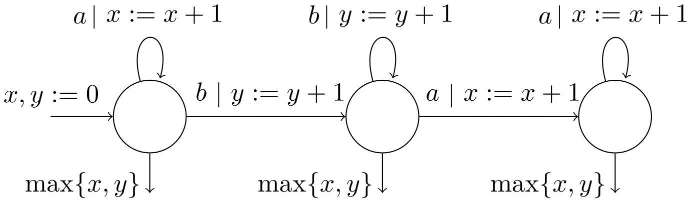
Plan A

The reverse f^r

$x : a's$
$y : b's$ in the last block



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Plan B

f using unambiguous nondeterminism

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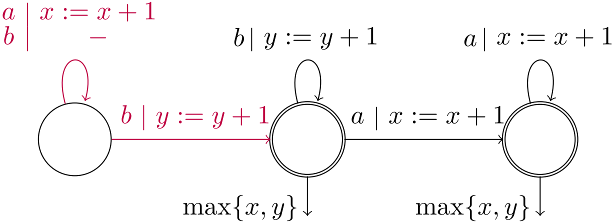

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Plan C

f using regular look-ahead

$x : a's$
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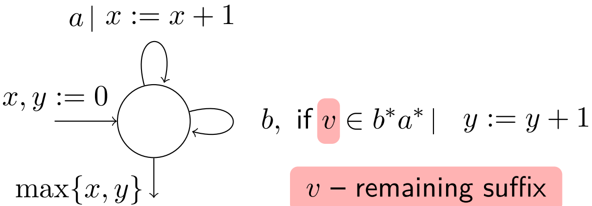

v – remaining suffix

BAC is a robust class!

One can define f without reverse, nondeterminism or regular look-ahead

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Copyless CRA vs WA

finitely ambiguous WA

WA

polynomially ambiguous WA

Copyless CRA vs WA

finitely ambiguous WA $\quad\quad\quad\quad$ WA $\subsetneq$ CRA

$\subsetneq \quad\quad\quad\quad \subsetneq$

polynomially ambiguous WA

Copyless CRA vs WA

$$
\begin{tikzcd}[column sep=small]
 & \text{BAC} \arrow[r, phantom, "\subsetneq"] & \text{copyless CRA} \arrow[dr, phantom, sloped, "\subsetneq"] & & \\
\text{finitely ambiguous WA} \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & & \text{WA} \arrow[r, phantom, "\subsetneq"] & \text{CRA} \\
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\end{tikzcd}
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Copyless CRA vs WA

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\begin{tikzcd}
& \text{BAC} \arrow[r, phantom, "\subsetneq"] & \text{copyless CRA} \arrow[dr, phantom, sloped, "\subsetneq"] \arrow[dd, phantom, "\not\subseteq"] & & \\
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\end{tikzcd}
$$

Copyless CRA vs WA

probably

$$
\begin{array}{ccccccc}
 & & \text{BAC} & \subsetneq & \text{copyless CRA} & & \\
 & \subsetneq & & & & \subsetneq & \\
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 & \subsetneq & & & & \subsetneq & \\
 & & & \text{polynomially ambiguous WA} & & &
\end{array}
$$

Maximal Partition logic

(a different approach)

Regular selectors

How to select intervals?

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With regular expressions

$$R\langle S\rangle T$$

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$$\underbrace{a\ b}_{\in R}\ \underbrace{b\ b}_{\in S}\ \underbrace{a\ a\ b\ b\ a\ b\ b\ b\ b\ a\ a\ b\ a}_{\in T}$$

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a | b | b | b | a a | b | b | a | b | b | b | b | a a | b | a

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a	b	b	b	a	a	b	b	a	b	b	b	b	a	a	b	a
	1	1	1			1	1		1	1	1	1			1	

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$1 + 1 + 1 \quad + \quad 1 + 1 \quad + \quad 1 + 1 + 1 + 1 \quad + \quad 1 \quad = \quad 10$

MP example

"The longest block of b's"

$$\mathrm{Max}\ \ b^{+}.\ \sum b.\ 1$$

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$$\color{green}{\mathrm{Max}\ b^+.}\ \sum b.\ 1$$

a | b b b | a a | b b | a | b b b b | a a | b | a

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$$\mathrm{Max}\ b^+.\ \sum b.\ 1$$

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b b b b b b b b b b

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$$\text{Max } b^+ .\ \sum b.\ 1$$

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| b | b | b | | b | b | | b | b | b | b | | b |

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a | b b b | a a | b b | a | b b b b | a a | b | a

| b | b | b | | b | b | | b | b | b | b | | b |

1 1 1   1 1   1 1 1 1   1

MP example

"The longest block of b's"

$\Sigma^*\langle b^+\rangle\Sigma^*$

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$$\mathrm{Max}\ b^+.\ \sum b.\ 1$$

a | b b b | a a | b b | a | b b b b | a a | b | a

| b | b | b | | b | b | | b | b | b | b | | b |

$1+1+1=3$ $1+1=2$ $1+1+1+1=4$ $1=1$

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For not maximal intervals MP $\not\subseteq$ WA

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| b | b | b | | b | b | | b | b | b | b | | b |

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all intervals

For not maximal intervals MP $\not\subseteq$ WA $\sum(\Sigma^*).1(w) = \mathcal{O}(|w|^2)$

Main result

Theorem

MP logic = BAC automata

Conclusions

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- What is decidable about copyless CRA?

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- What is decidable about copyless CRA?
- BAC = MP logic
- How are full CRA related with full WMSO?