The boundedness and zero isolation problems for weighted automata over nonnegative rationals

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Move seminar, Marseille

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Outline

1. Introduction: weighted automata over positive rationals

2. Boundedness

3. Zero-isolation

Weighted automata (WA)

Domain: $\mathbb{Q}_{\geq 0}$ (these generalise probabilistic automata)

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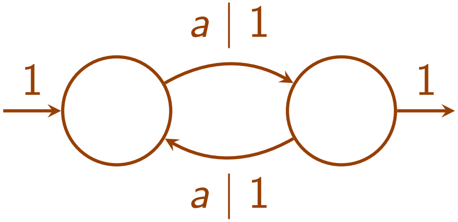
- Example 1: $\mathcal{A}$



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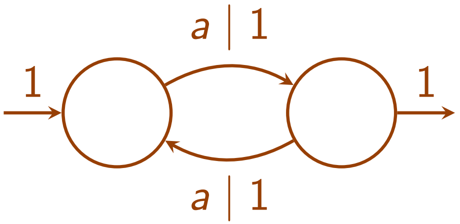

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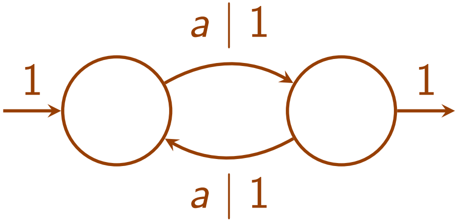

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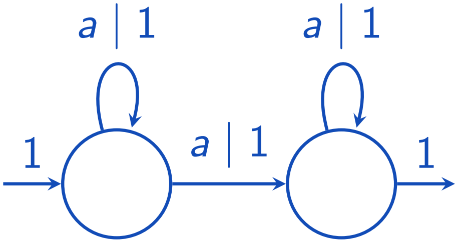

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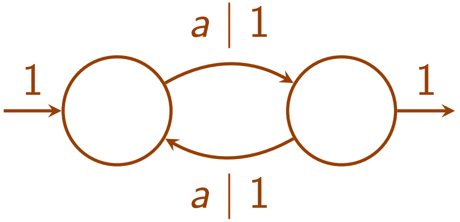

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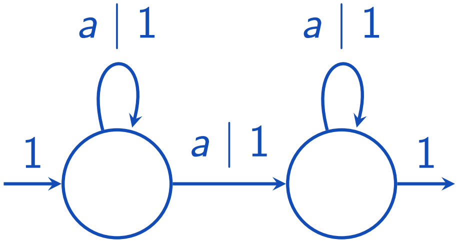

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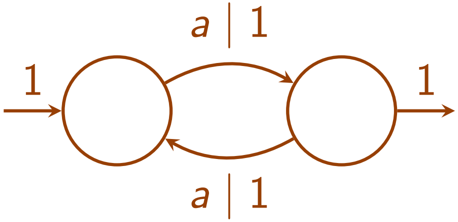

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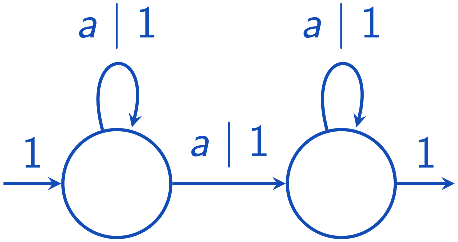

Output: $\mathcal{B}(a^n) = n$

Matrix definition

$$\mathcal{A}(a^n) = (1,0)\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}^n\begin{pmatrix}0\\1\end{pmatrix}, \quad \mathcal{B}(a^n) = (1,0)\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}^n\begin{pmatrix}0\\1\end{pmatrix}$$

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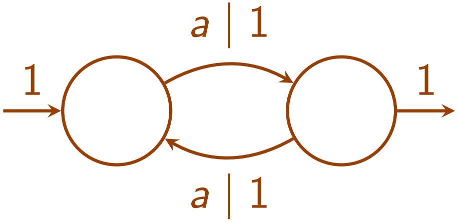

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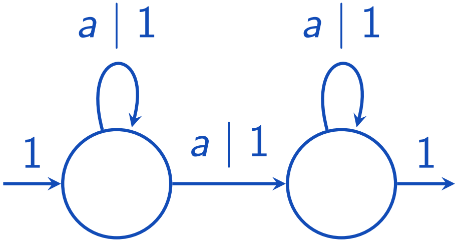

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In general $\mathcal{A}(abba) = I^\top M_a M_b M_b M_a F = I^\top M_{abba} F$

Cost-register automata (CRA)

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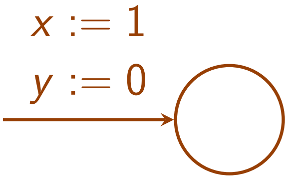

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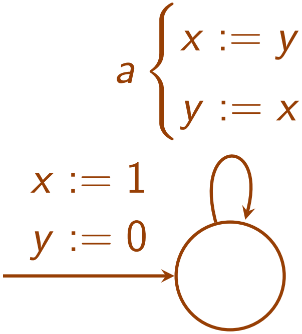

$$a\begin{cases}x := y\\ y := x\end{cases}$$

$x := 1$

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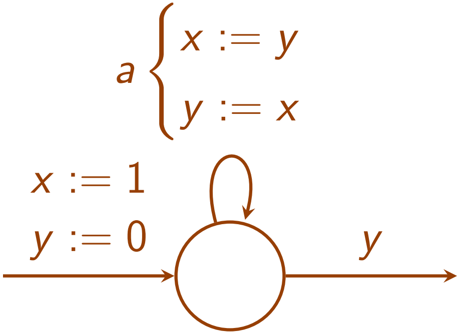

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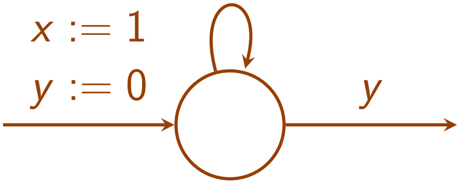

- $\mathcal{B}$ as a CRA

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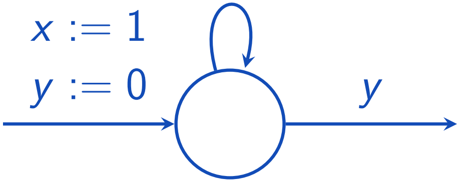

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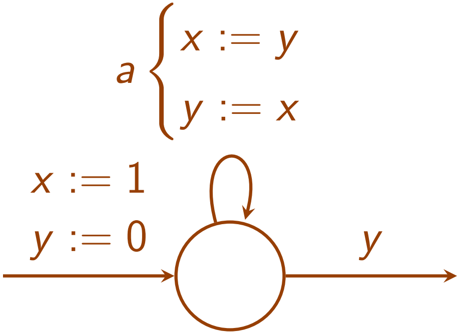

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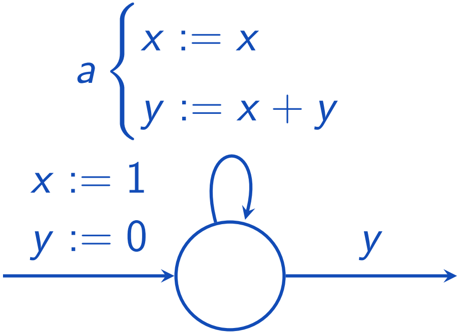

Linear CRA = WA (CRA are nonlinear in general)

Subclasses of weighted automata and CRA

1. (WA) ambiguity: bound on accepting runs

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Notation:

- $A \subseteq B$: for all (commutative) semirings A is contained in B
- $A \not\subseteq B$: there exists a (commutative) semiring s.t. A is not contained in B

Subclasses compared

$$\begin{tikzcd}
\text{fin-amb WA} \arrow[d, phantom, "\not\supseteq \quad \not\subseteq"] & & \text{poly-amb WA} \arrow[dr, phantom, "\subsetneq"] & & & \\
 & \subsetneq & & \text{WA} & = & \text{linear CRA} \\
\text{copyless linear CRA} & & \text{copyless CRA} \arrow[ur, phantom, "\subsetneq"] & & &
\end{tikzcd}$$

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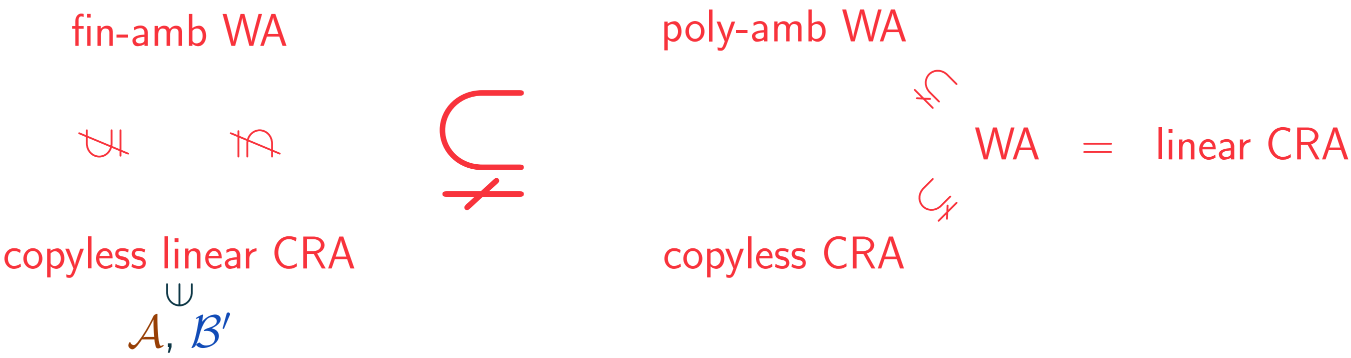

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$\Cup$

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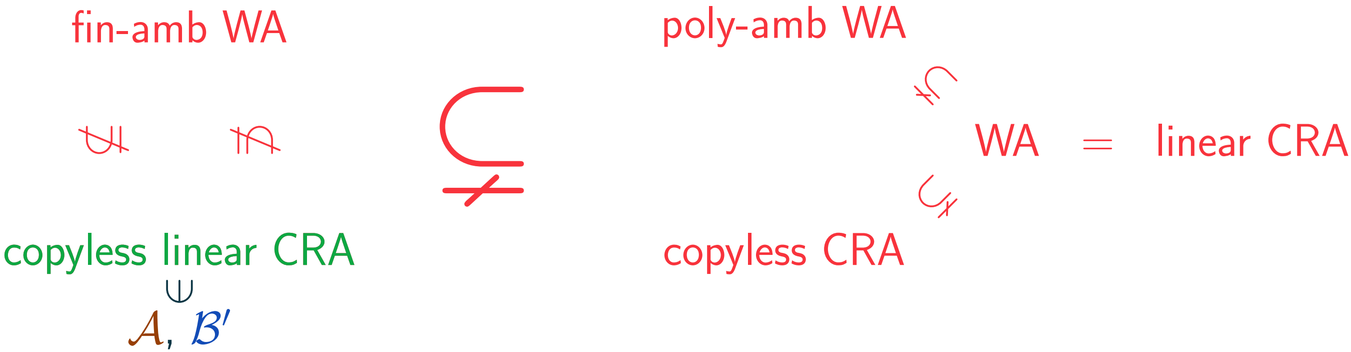

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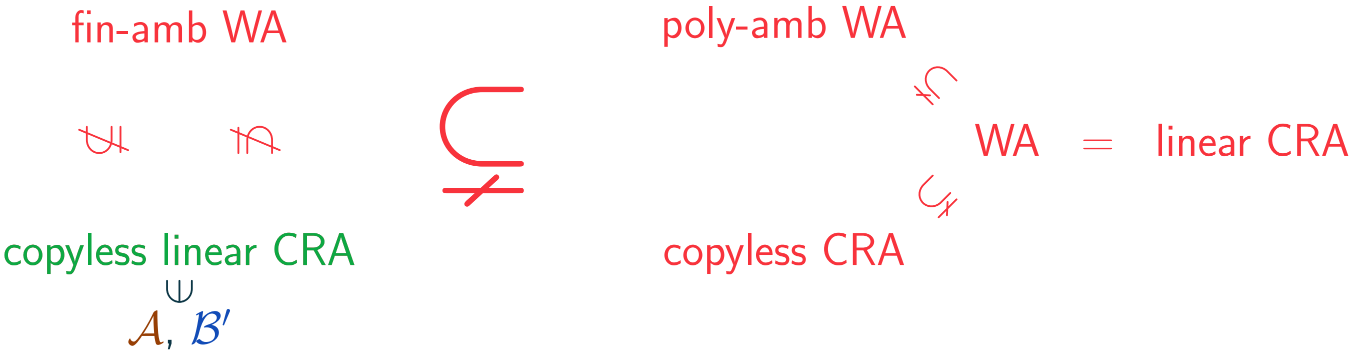

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$\in$

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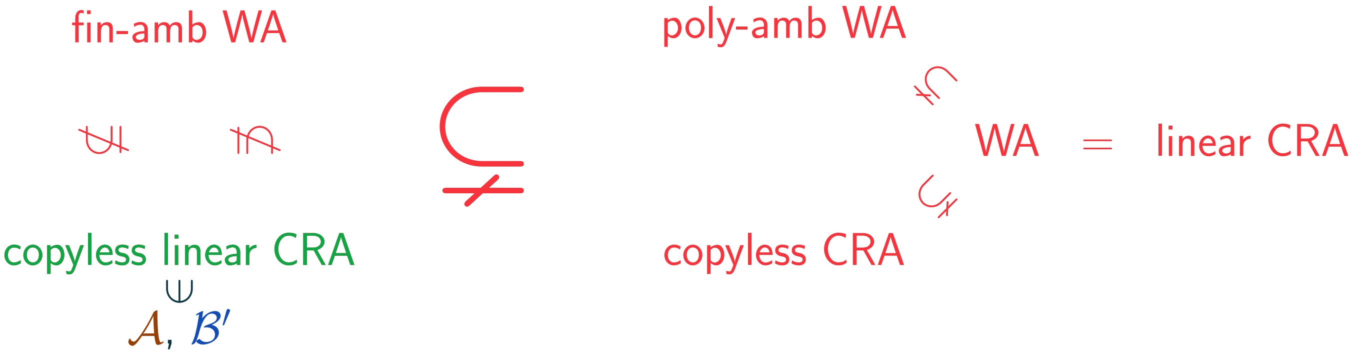

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- They're not [Almagor et al. 2018]

  but the class is interesting

Threshold problems

Given WA $\mathcal{A}$ and $c \in \mathbb{Q}_{\geq 0}$:

Is $\mathcal{A}(w) \geq c$ (or $\mathcal{A}(w) \leq c$) for all $w \in \Sigma^*$?

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$\leq$-threshold and $\geq$-threshold are "the same" for probabilistic automata

$\mathcal{A}(w) \leq \frac{1}{2} \quad \text{iff} \quad 1 - \mathcal{A}(w) \geq \frac{1}{2}$.

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Theorem (Daviaud et al. 2018)

For finitely-ambiguous WA over $\mathbb{Q}_{\geq 0}$

- $\leq$-threshold is trivially decidable
- $\geq$-threshold nontrivially decidable (subject to Schanuel's conjecture)

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Theorem (our result)

Boundedness is decidable for linear copyless CRA in polynomial time.

Patterns

How can a WA $\mathcal{A}$ be unbounded?

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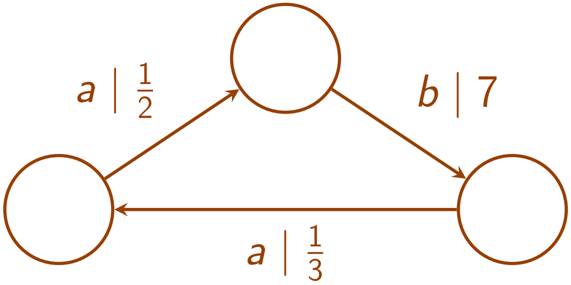
- First pattern

  is there a loop of value > 1?

  $aba \mid \frac{7}{6}$



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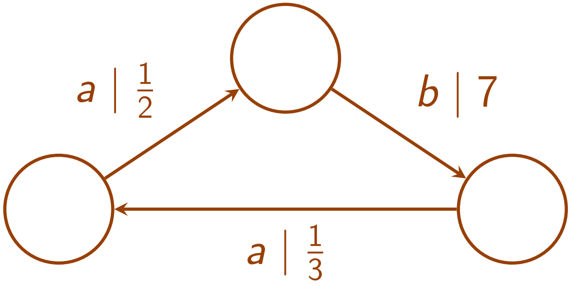

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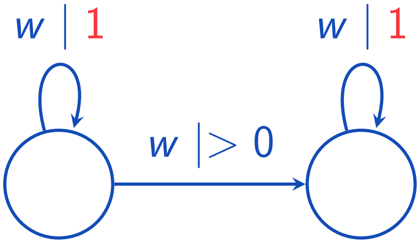

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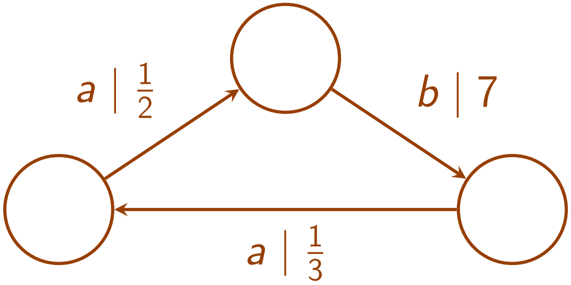

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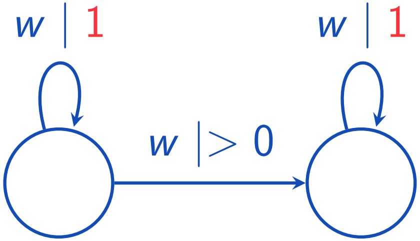

Inspired by patterns characterising ambiguity [Weber and Seidl, 1991]

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Theorem (main contribution)

A copyless linear CRA $\mathcal{A}$ is bounded iff the previous patterns don't occur.

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- Checking patterns: reachability questions in weighted graphs (e.g. Dijkstra).

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  - if there are many (polynomially) their value becomes small (exponentially)

Number of runs vs their value

Lemma

Suppose $\mathcal{A}$ without patterns. For every natural $k \geq 2$ and $w \in \Sigma^*$:

the number of runs of value $> \frac{1}{k}$ is $\operatorname{poly}\log k$ (does not depend on $|w|$)

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Remains to prove the lemma

Copyless linear CRA as WA

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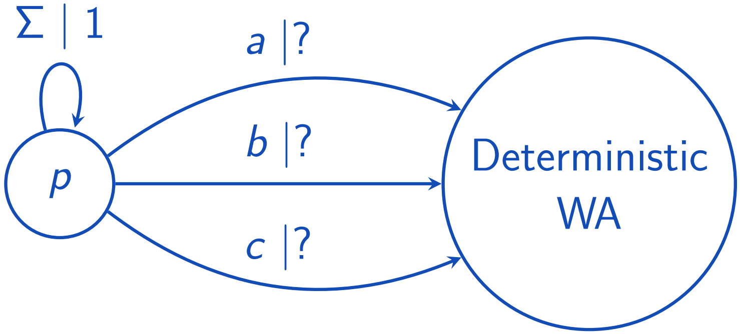
For boundedness: copyless linear CRA can be reduced to WA of the form:



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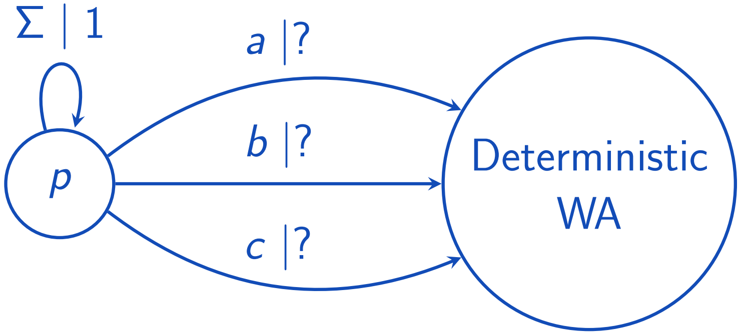

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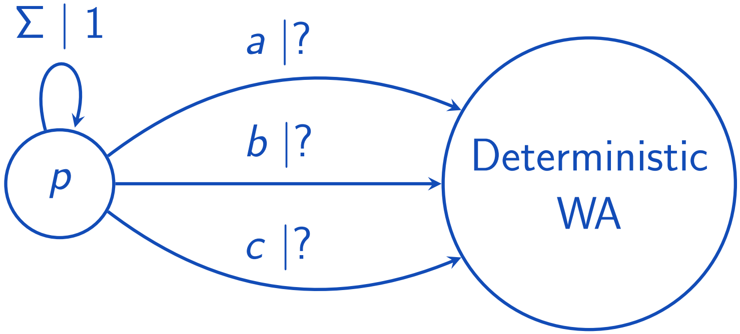

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For the rest of this part we work with this model.

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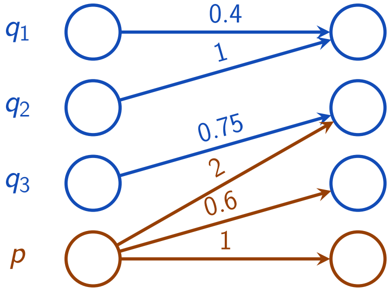

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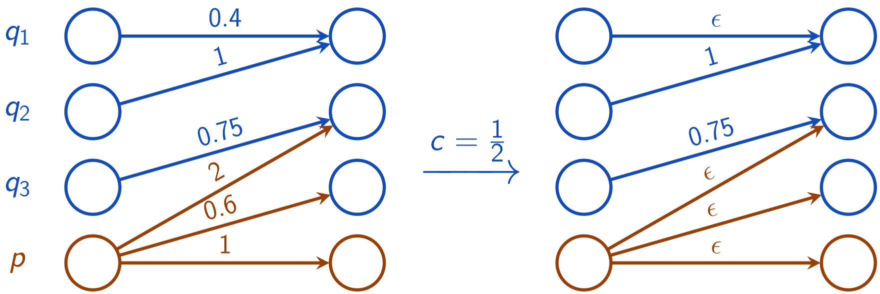

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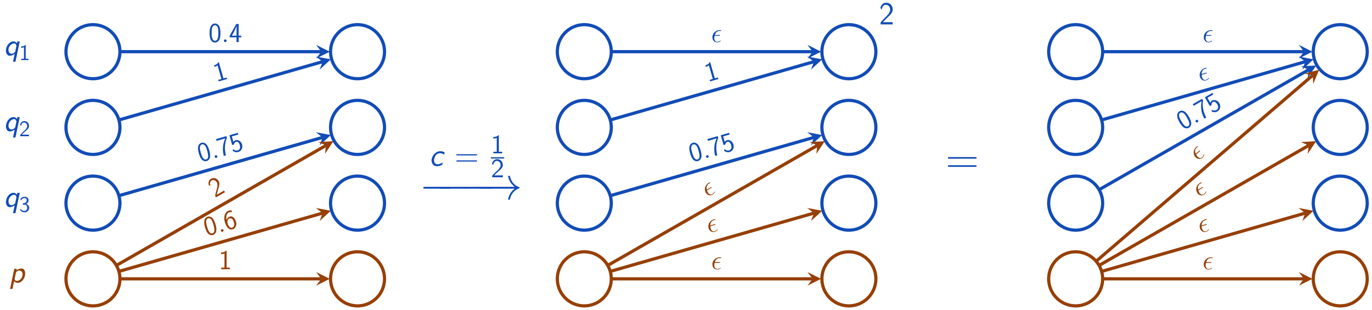

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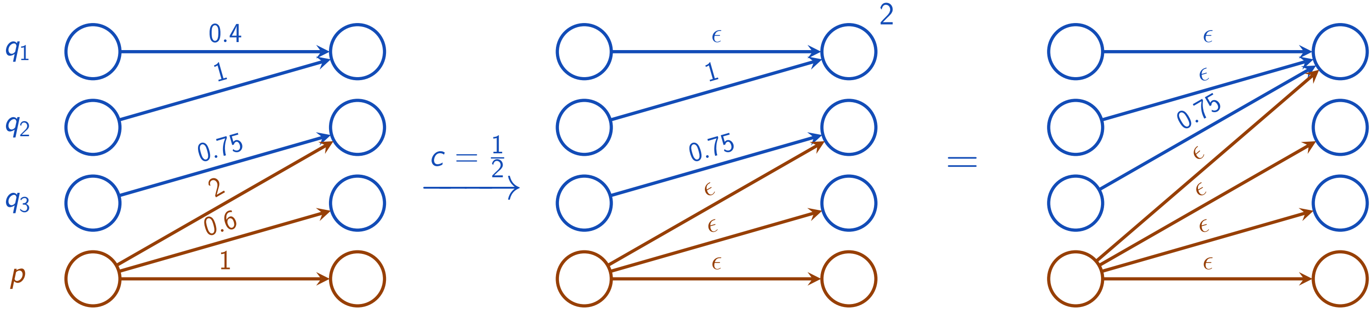

We get a finite monoid $\{\overline{M}_w \mid w \in \Sigma\}$ (with the matrix product).

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Let $s_1, \ldots, s_n \in \mathcal{M}$

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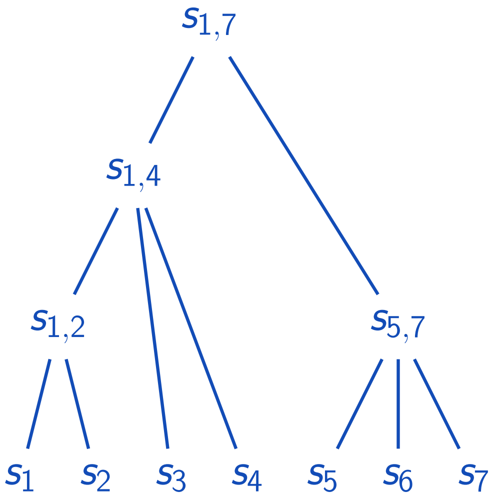

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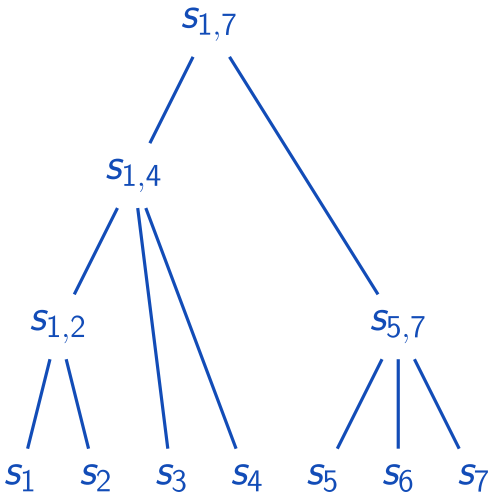

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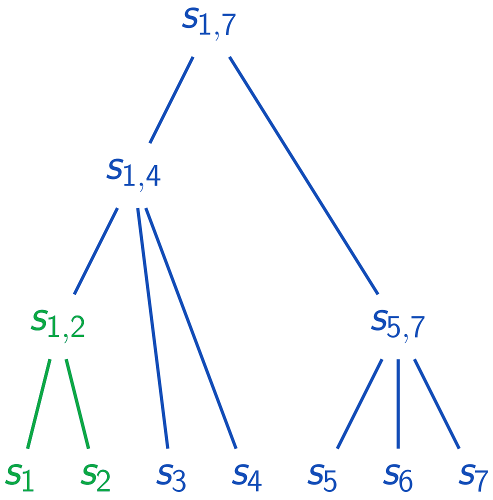

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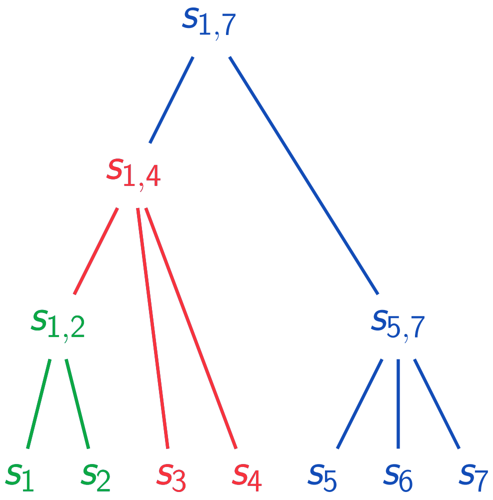

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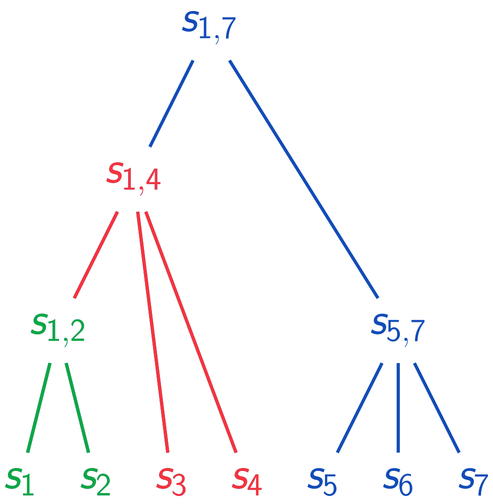

Theorem (Simon 1990)

There is always a factorisation of height at most $9|\mathcal{M}|$.

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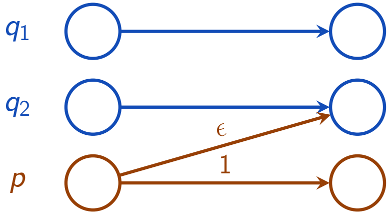

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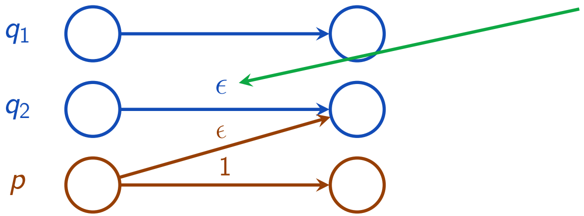

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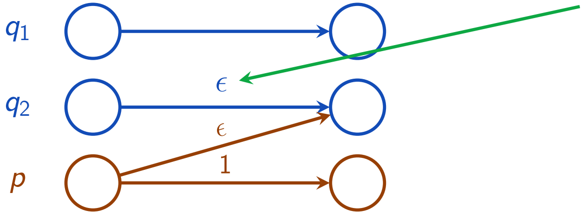

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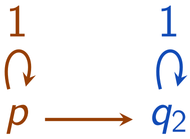

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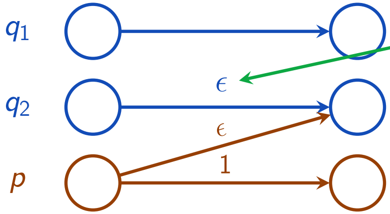

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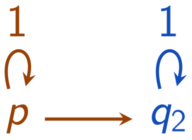

Since $\epsilon^{\log k} < \frac{1}{k}$

number of runs $\leq$ $\log k \cdot \mathcal{O}(\log^i k)$

Outline

1. Introduction: weighted automata over positive rationals

2. Boundedness

3. Zero-isolation

The zero isolation problem

Given $\mathcal{A}$ over $\mathbb{Q}_{\geq 0}$:

is there $c \in \mathbb{Q}_{\geq 0} \setminus \{0\}$ such that $\mathcal{A}(w) \geq c$ for all $w \in \Sigma^*$?

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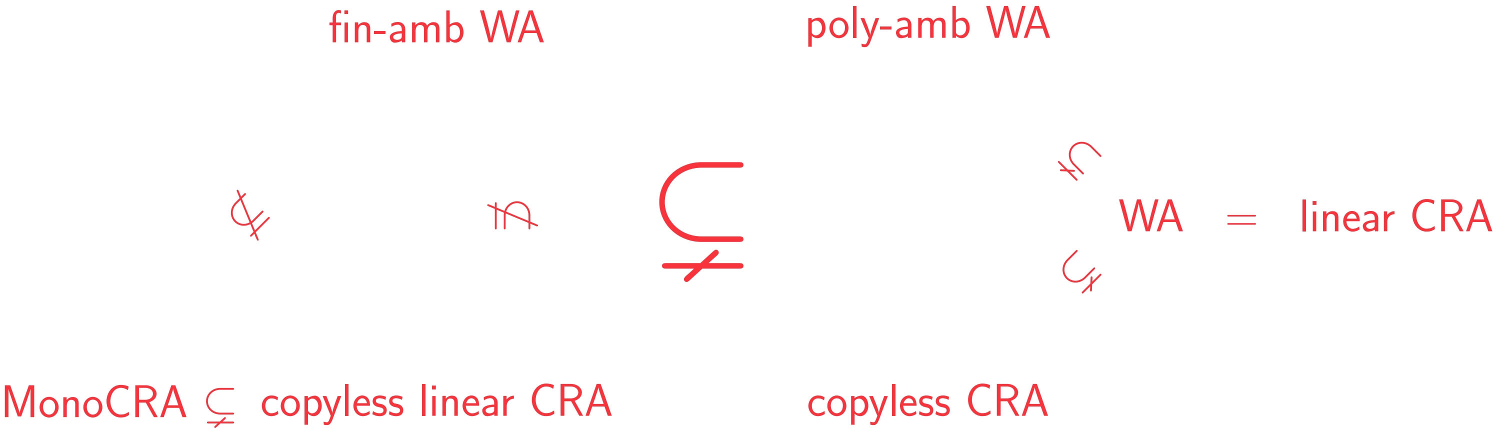
Theorem (our result)

Zero isolation is decidable for MonoCRA (to be defined) subject to Schanuel's conjecture.

Results summary


fin-amb WA

poly-amb WA

$\not\subseteq$ $\not\supseteq$ $\subsetneq$ $\subsetneq$

WA $=$ linear CRA

$\subsetneq$

MonoCRA $\subsetneq$ copyless linear CRA

copyless CRA

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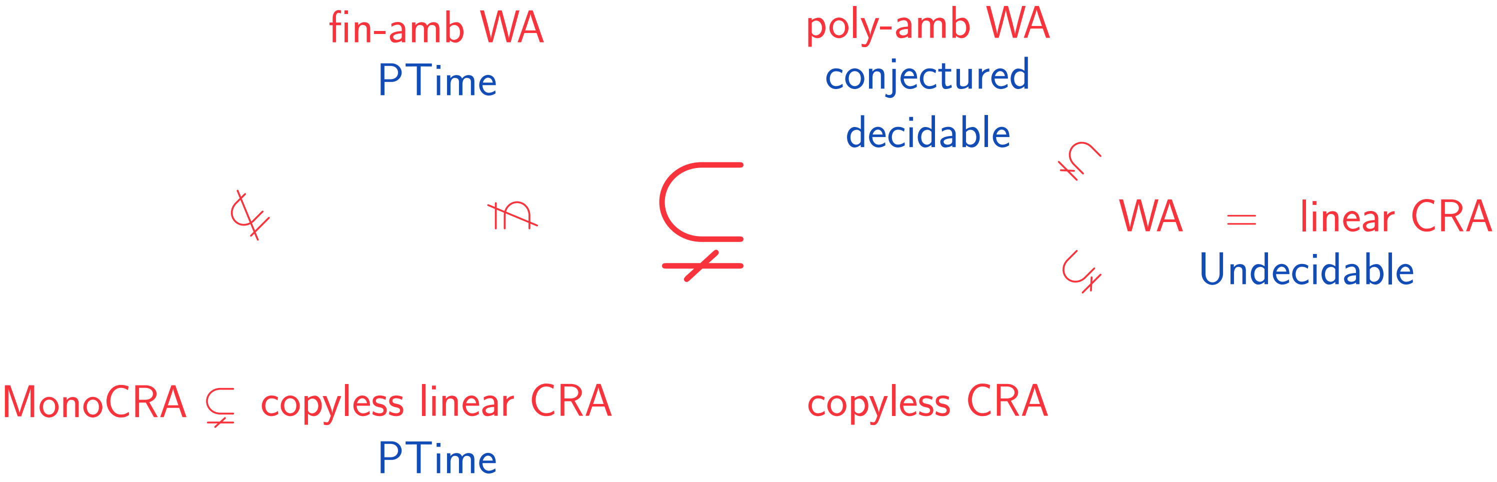

fin-amb WA
PTime

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conjectured
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- Boundedness

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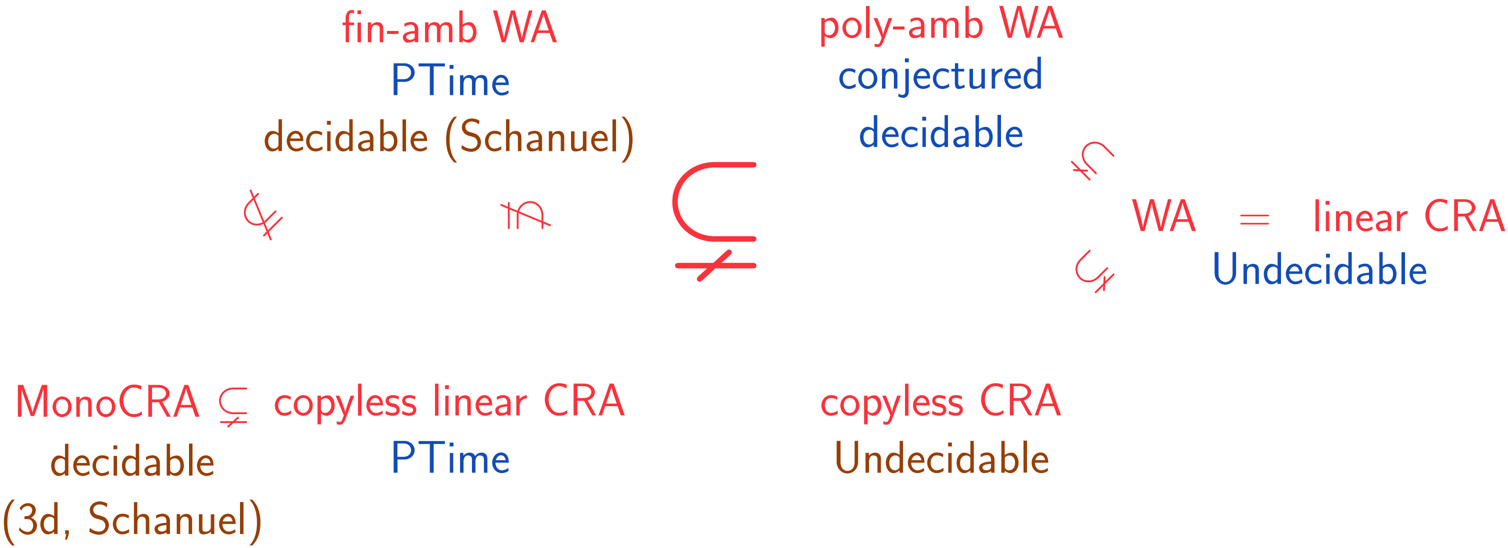

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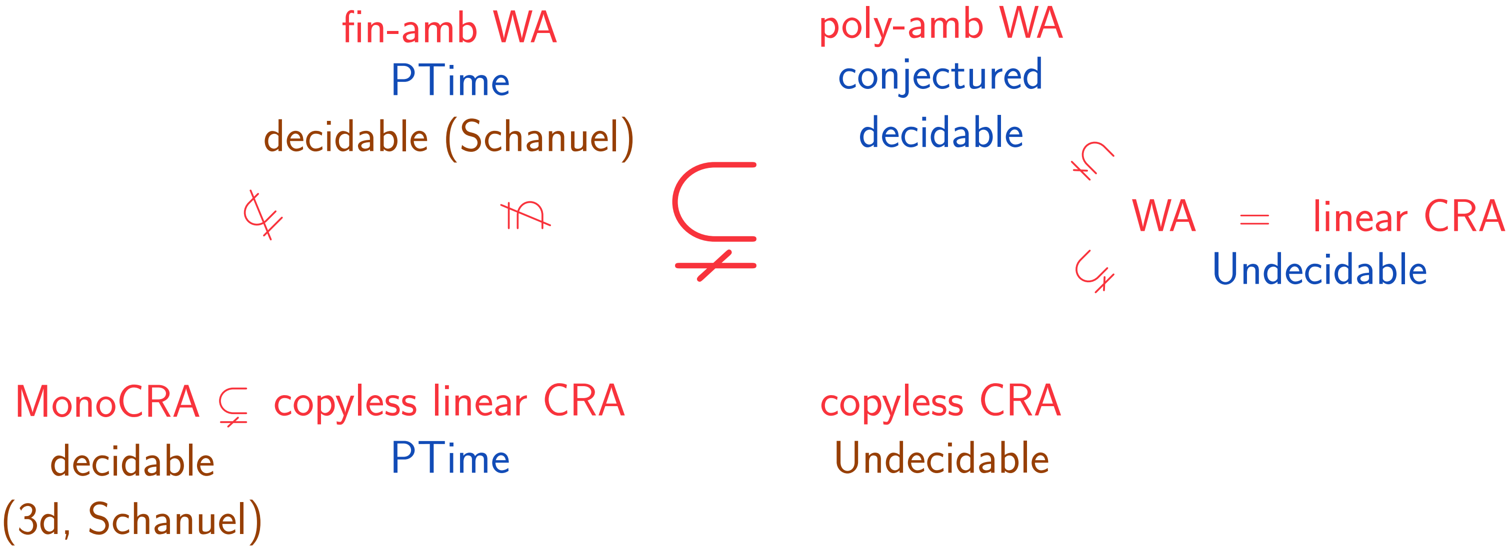

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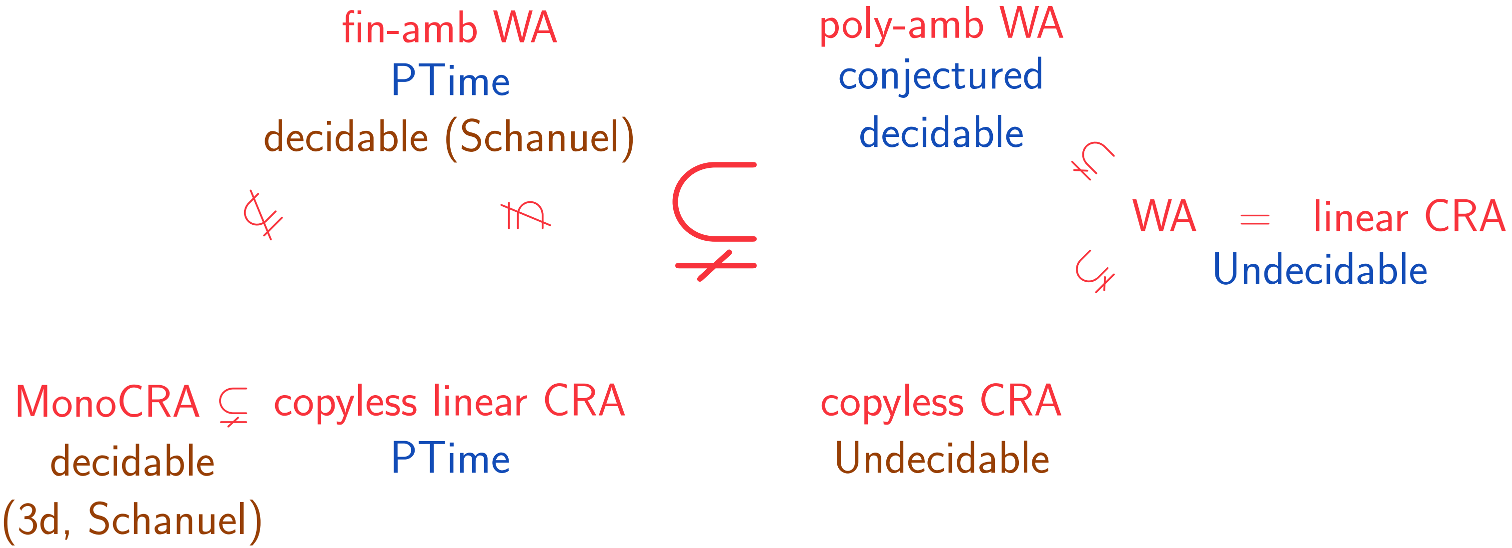

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This is just a syntactic translation if $\mathbb{Q}(+, \cdot)$ is changed to $\mathbb{Q}(\max, \cdot)$.

Orthant vector addition systems (OVAS)

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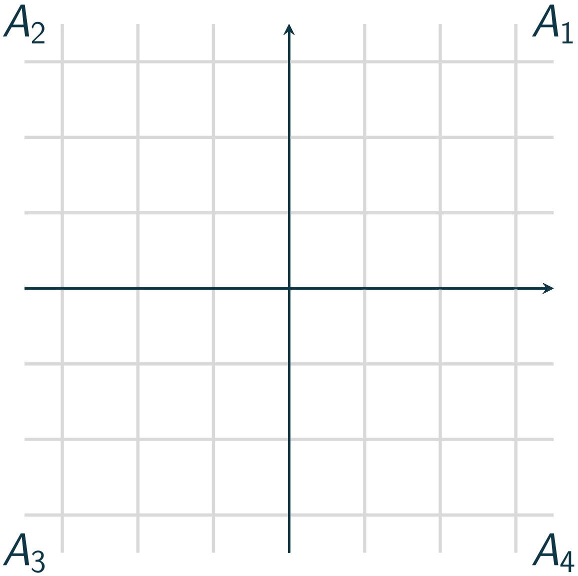

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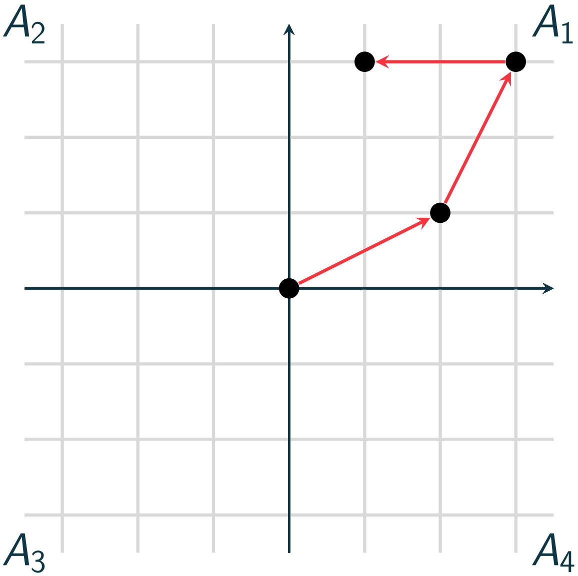

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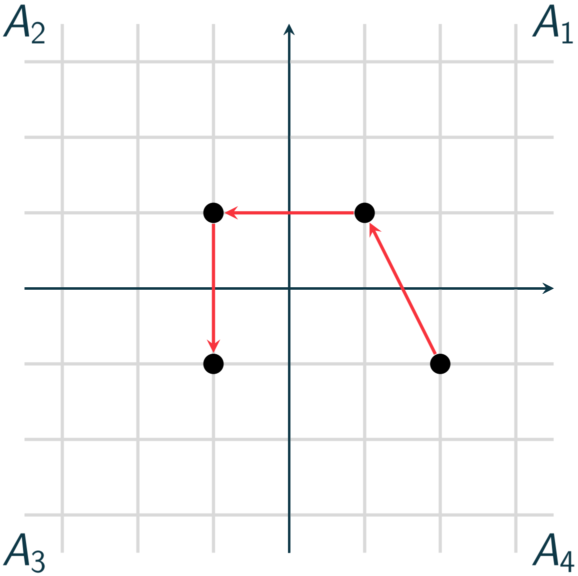

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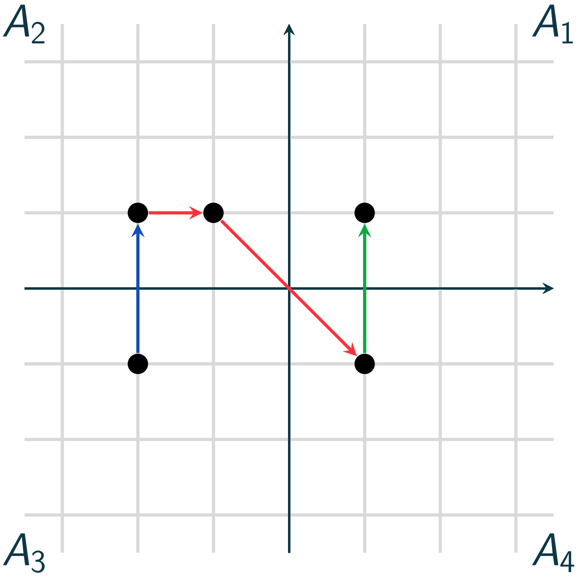

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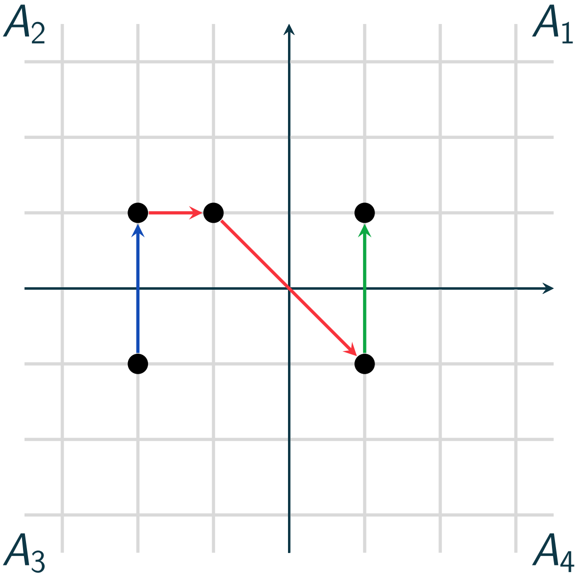

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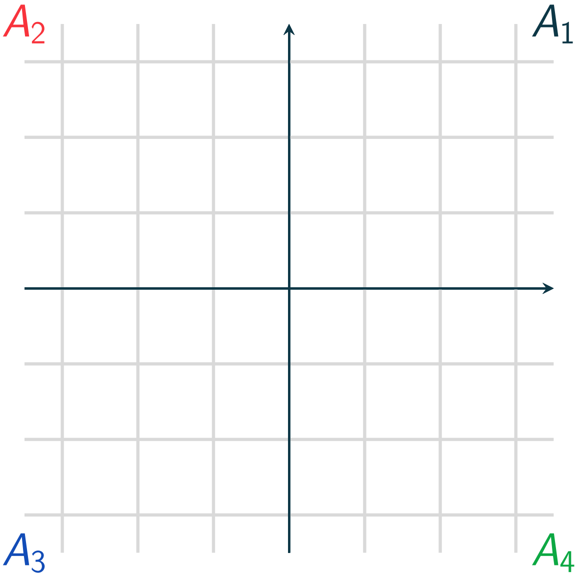
Coverability problems for OVAS

- Coverability: given OVAS, point p is there a path to the positive orthant?



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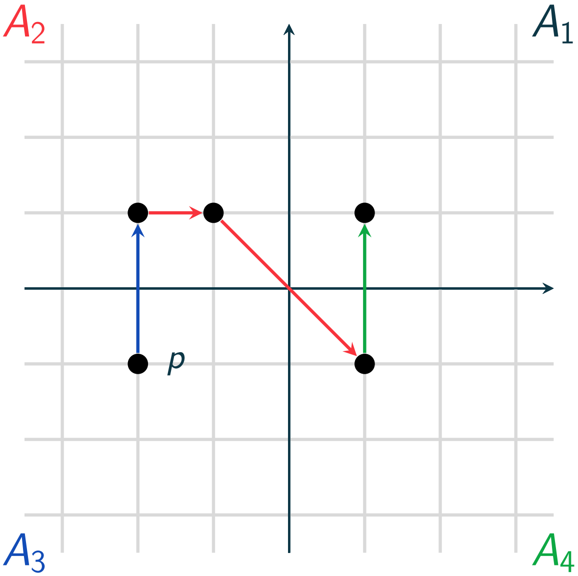

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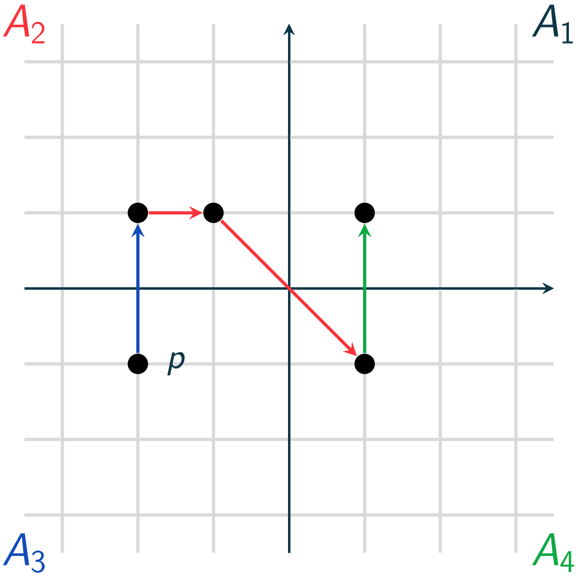

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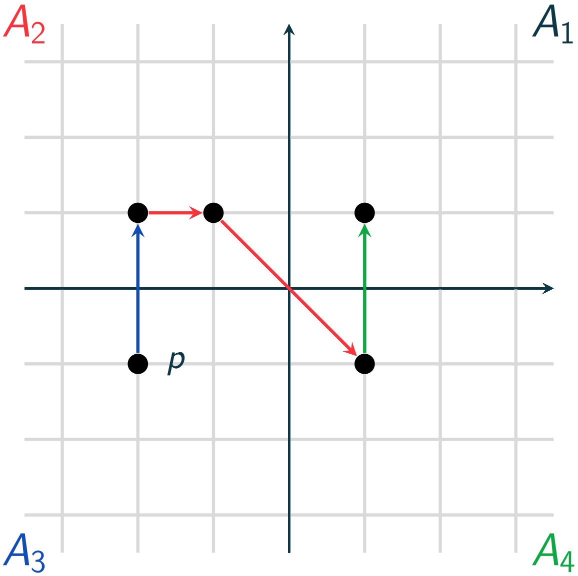

Theorem

Coverability is undecidable. Universal coverability is decidable in dimension 3.

OVAS and MonoCRA

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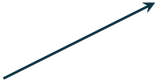

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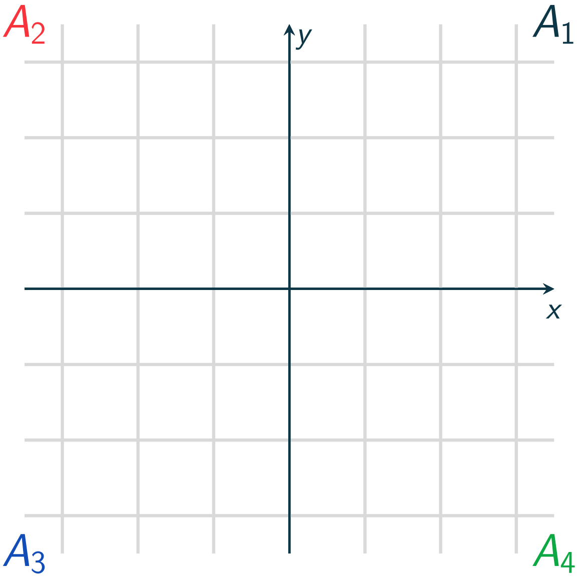

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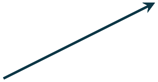

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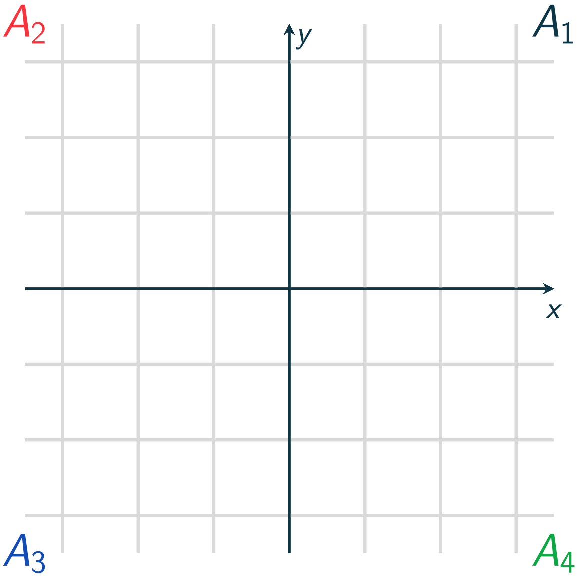

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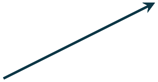

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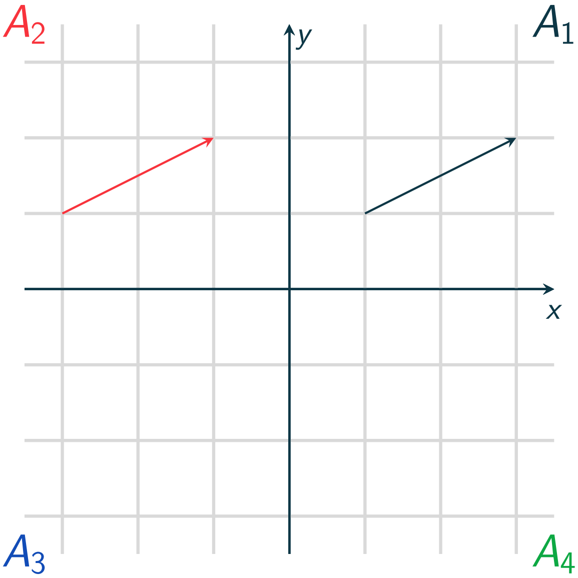

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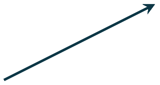

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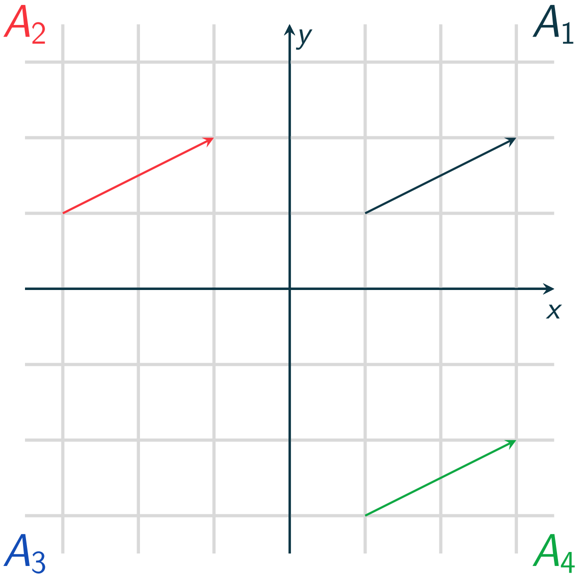

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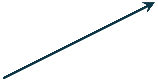

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  Intuition: if $\mathcal{A}(w\sigma w')$ is big then

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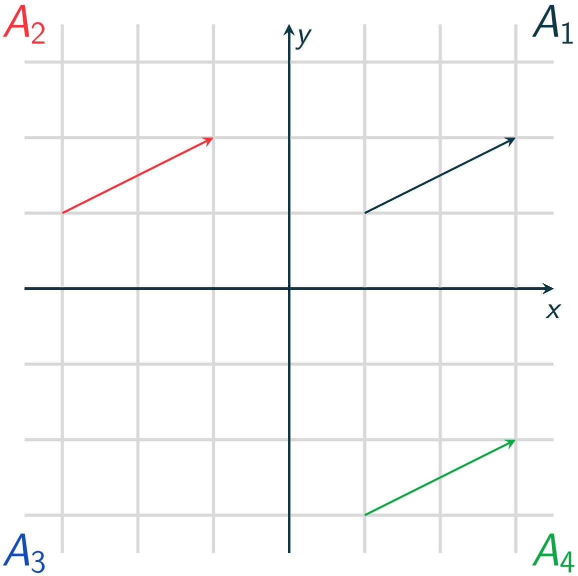

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So the MonoCRA class is not that trivial.

Universal coverability for 3-OVAS

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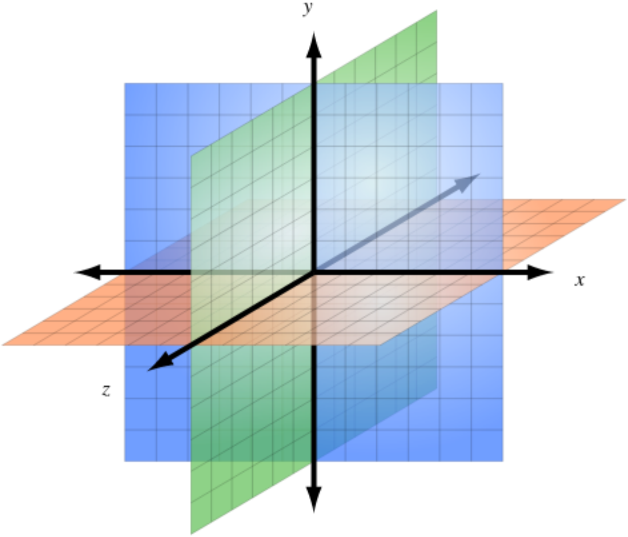

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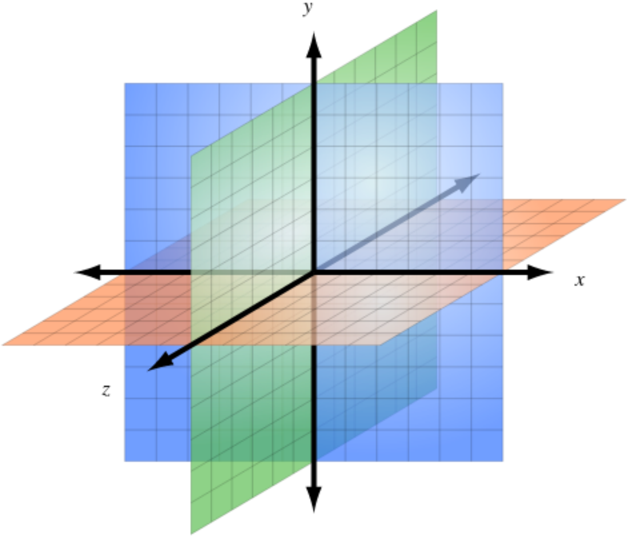

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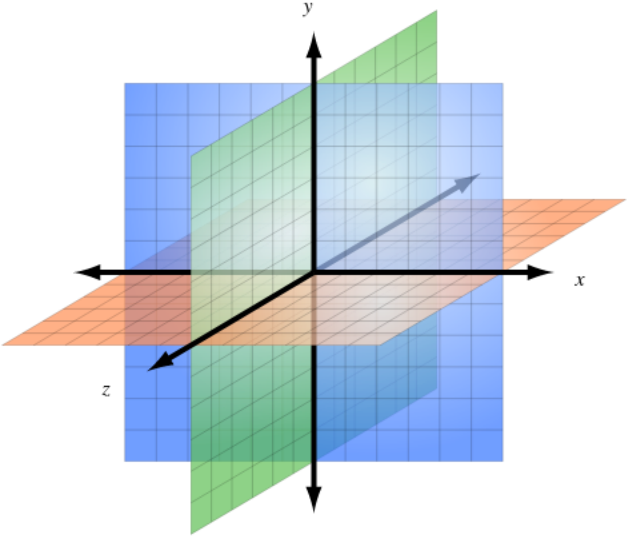

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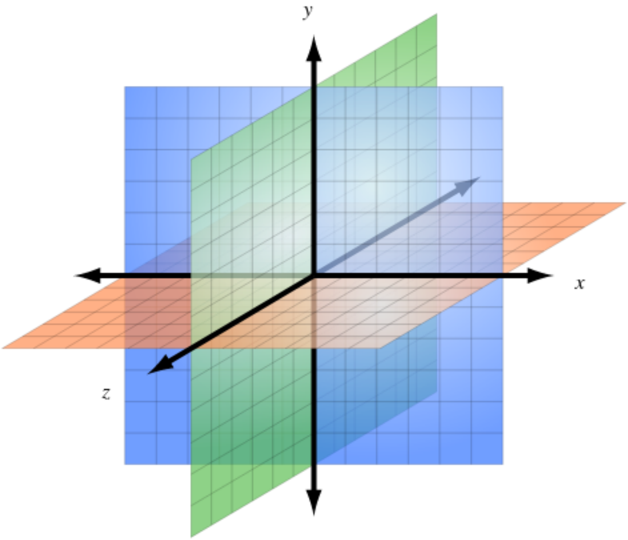

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- Many open problems left :)

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