On polynomial recursive sequences

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Outline

1. Introduction (mostly linear recursive sequences)

2. Polynomial recursive sequences

3. Proof that n^n is not polynomially recursive

4. Applications in weighted automata

Recursive sequences

- **Fibonacci sequence** F_n

$$f_0 = 0, \qquad f_1 = 1, \qquad f_n = f_{n-1} + f_{n-2} \quad \text{for} \quad n \geq 2$$

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}
0	1	1	2	3	5	8	13	21	34	55	89

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- **Catalan numbers** C_n

$$C_0 = 1, \qquad C_1 = 1, \qquad C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i} \quad \text{for} \quad n \geq 2$$

C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7
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Linear recursive sequences (over $\mathbb{Q}$)

$$u_{n+k} = a_k u_{n+k-1} + a_{k-1} u_{n+k-1} + \ldots + a_1 u_n$$

fixing $a_1, \ldots, a_k \in \mathbb{Q}$ and $u_0, \ldots, u_{k-1} \in \mathbb{Q}$

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Definition

u_n is linear recursive if there is $L(x_1, x_2, \ldots, x_k) = a_1 x_1 + \ldots a_k x_k$

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We will see that **Catalan numbers** C_n are not linear recursive

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$$\vec{u_n}^{\mathsf{T}} = M^n \cdot I$$

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$$\vec{u_n} = (u_n^1, \ldots, u_n^k)$$

$$I = (c_1, \ldots, c_k)$$

$$M[i, \bullet] = L_i$$

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a_n is linear recursive iff a_n is definable as a linear system

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For depth k use $k-1$ auxiliary shifts.

$$a_{n+4} = 3a_{n+3} - 2a_{n+2} + 4a_{n+1} - a_n \quad \rightsquigarrow \quad \begin{cases} a_{n+1} = b_n \\ b_{n+1} = c_n \\ c_{n+1} = d_n \\ d_{n+1} = 3a_n - 2b_n + 4c_n - d_n \end{cases}$$

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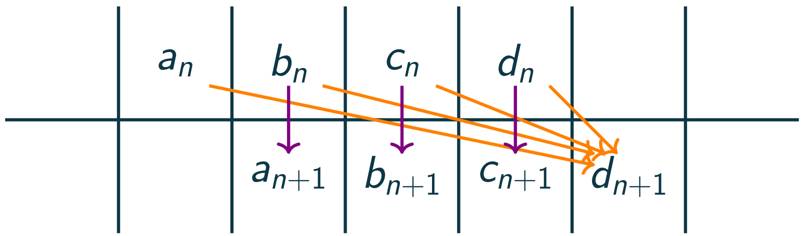

From systems to one linear recursive sequence

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$$M = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad det(M - \lambda I) = -\lambda^3 + 3\lambda^2 - 3\lambda + 1$$

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Then $a_{n+3} = 3a_{n+2} - 3a_{n+1} + a_n$ (by the Cayley–Hamilton theorem)

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$K(x_0, \ldots, x_k) = a_0 x_0 + \ldots a_k x_k \quad u_{n+k} = -\frac{a_{k-1}}{a_k} u_{n+k-1} - \ldots - \frac{a_0}{a_k} u_n$

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Definition

A sequence a_n is **polynomial recursive** if there are $u_n^1, \ldots, u_n^k$, $a_n = u_n^1$ and polynomials $P_1, \ldots, P_k \in \mathbb{Q}[x_1, \ldots, x_k]$

$$\begin{cases} u_0^1 = c_1 \\ u_0^2 = c_2 \\ \quad \vdots \\ u_0^k = c_k \end{cases} \qquad \begin{cases} u_{n+1}^1 = P_1(u_n^1, u_n^2, \ldots, u_n^k) \\ u_{n+1}^2 = P_2(u_n^1, u_n^2, \ldots, u_n^k) \\ \quad \vdots \\ u_{n+1}^k = P_k(u_n^1, u_n^2, \ldots, u_n^k) \end{cases}$$

Polynomial recursive sequences

Example: $a_n = n!$ $\qquad b_n = n+1$

$$\begin{cases} a_0 = 1 \\ b_0 = 1 \end{cases} \qquad \begin{cases} a_n = a_{n-1} \cdot b_{n-1} \\ b_n = b_{n-1} + 1 \end{cases}$$

Remark: $n! \approx 2^{n \log(n)}$ is not linear recursive.

u_n linear recursive $\implies |u_n| \leq C^n$ for some $C \in \mathbb{Q}$

generalise linear recursive sequences

Definition

A sequence a_n is **polynomial recursive** if there are $u_n^1, \ldots, u_n^k$, $a_n = u_n^1$ and polynomials $P_1, \ldots, P_k \in \mathbb{Q}[x_1, \ldots, x_k]$

$$\begin{cases} u_0^1 = c_1 \\ u_0^2 = c_2 \\ \vdots \\ u_0^k = c_k \end{cases} \qquad \begin{cases} u_{n+1}^1 = P_1(u_n^1, u_n^2, \ldots, u_n^k) \\ u_{n+1}^2 = P_2(u_n^1, u_n^2, \ldots, u_n^k) \\ \vdots \\ u_{n+1}^k = P_k(u_n^1, u_n^2, \ldots, u_n^k) \end{cases}$$

Polynomial recursive sequences examples

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2. Fact. C_n is odd iff $n = 2^m - 1$ for some m

∎

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Polynomial recursive sequences with one polynomial

Definition

Consider a class of sequences defined by

$$u_{n+k} = P(u_n, \dots, u_{n+k-1})$$

for some $P \in \mathbb{Q}[x_1, \dots, x_k]$.

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$P(x_1, x_2, x_3) = x_3 x_1 - x_2^2 - x_2 x_1$

$$P(u_n, u_{n+1}, u_{n+2}) = 0$$

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u_{n+k} nonlinear

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A sequence u_n admits a cancelling polynomial if

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Remark

The converse is not true.

$$P(x_1) = x_1^2 - 1$$

is cancelling for any u_n over $\{-1, 1\}$

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If u_n is polynomial recursive then it admits a cancelling polynomial.

Proof. (Similar to the less constructive proof)

$$\begin{cases} u^1_{n+1} = P_1(u^1_n, u^2_n, \ldots, u^k_n) \\ u^2_{n+1} = P_2(u^1_n, u^2_n, \ldots, u^k_n) \\ \qquad\vdots \\ u^k_{n+1} = P_k(u^1_n, u^2_n, \ldots, u^k_n) \end{cases}$$

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   so there is $Q \in \mathbb{Q}[y_0, \ldots, y_k]$ s.t. $Q(P_1^{(0)}, \ldots, P_1^{(k)}) = 0$ ∎

Outline

1. Introduction (mostly linear recursive sequences)

2. Polynomial recursive sequences

3. **Proof that n^n is not polynomially recursive**

4. Applications in weighted automata

Failed attempts

Theorem

$u_n = n^n$ is not polynomial recursive

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1. Asymptotic argument?

Recall that $n!$ is polynomial recursive

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So is $3^n n!$

$n! \leq n^n \leq 3^n n!$ (Stirling)

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Recall that polynomial recursive sequences are ultimately periodic $\mod p$

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2. Ultimate periodicity $\mod p$? **No!**

Recall that polynomial recursive sequences are ultimately periodic $\mod p$

$u_{n+p(p-1)} \equiv u_n \mod p$ (little Fermat)

Proof using cancelling polynomials

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- If u_n is polynomial recursive there is a cancelling polynomial $Z \in \mathbb{Q}[x_0, \ldots, x_k]$

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$$P(x) = 4(x+1)^2(x+2)^6,$$

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Conclusion: Rewrite $Z(n^n, \ldots, (n+k)^{n+k}) = \sum_{i=1}^{\ell} P_i(n) \cdot Q_i(n)^n$,

where $P_i, Q_i \in \mathbb{Z}[x]$ are nonzero, and Q_i are pairwise different.

Cancelling polynomial for n^n

It remains to show (by contradiction)

Lemma

There are no $P_1, \ldots, P_\ell, Q_1, \ldots, Q_\ell \in \mathbb{Z}[x]$, where Q_i pairwise different and

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$$\sum_{i=1}^{\ell} P_i(a) \cdot Q_i(a)^b \equiv 0 \mod p \qquad \text{for all } a, b \in \mathbb{Z}\ (b > 0),\ p \text{ prime}.$$

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$$n \equiv a \mod p \qquad \Rightarrow \qquad P_i(n) \equiv P_i(a) \bmod p$$

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$$\sum_{i=1}^{\ell} P_i(a) \cdot Q_i(a)^b \equiv 0 \mod p \qquad \text{for all } a, b \in \mathbb{Z}\ (b > 0),\ p \text{ prime.}$$

Proof. Take $n \in \mathbb{N}$ such that $n \equiv_p a$ and $n \equiv_{p-1} b$ (p and $p-1$ are coprime)

$$n \equiv a \mod p \qquad \Rightarrow \qquad P_i(n) \equiv P_i(a) \bmod p$$

(little Fermat)

$$n \equiv b \mod p-1,\ b > 0 \qquad \Rightarrow \qquad Q_i(n)^n \equiv Q_i(a)^n \equiv Q_i(a)^b \bmod p$$

Cancelling polynomial for n^n

It remains to show (by contradiction)

Lemma

There are no $P_1, \dots, P_\ell, Q_1, \dots, Q_\ell \in \mathbb{Z}[x]$, where Q_i pairwise different and

$$\sum_{i=1}^{\ell} P_i(n) \cdot Q_i(n)^n = 0 \qquad \text{for all } n \in \mathbb{N}.$$

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P_1(a) \cdot Q_1(a)^3 + \ldots + P_\ell(a) \cdot Q_\ell(a)^3 &\equiv 0 \mod p\\
&\vdots\\
P_1(a) \cdot Q_1(a)^\ell + \ldots + P_\ell(a) \cdot Q_\ell(a)^\ell &\equiv 0 \mod p
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Outline

1. Introduction (mostly linear recursive sequences)

2. Polynomial recursive sequences

3. Proof that n^n is not polynomially recursive

4. **Applications in weighted automata**

Linear recursive sequences and weighted automata

Definition

A weighted automaton $\mathcal{A}$ over $\mathbb{Q}$ is $(d, \Sigma, \{M_a\}_{a\in\Sigma}, I, F)$, where:

- $d \in \mathbb{N}$ is the dimension;
- Σ is a finite alphabet;
- every M_a is a $d \times d$ matrix over $\mathbb{Q}$;
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- If $|\Sigma| = 1$ then $\Sigma^* = \{\epsilon, a, a^2, \ldots\} \equiv \mathbb{N}$

  Then $[\![\mathcal{A}]\!] : \mathbb{N} \to \mathbb{Q}$ are sequences

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  Then $[\![\mathcal{A}]\!] : \mathbb{N} \to \mathbb{Q}$ are sequences
- If $|\Sigma| = 1$ there is only one $M_a = M$

$$[\![\mathcal{A}]\!](n) = I^\top \cdot M^n \cdot F \quad \text{(linear recursive sequences)}$$

Cost-register automata

Definition

Cost-register automata over $\mathbb{Q}$ are polynomial recursive sequences

like weighted automata over $\mathbb{Q}$ are linear recursive sequences

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Fun fact: this model was defined (at least) 3 times

- polynomial recurrent relations [Sénizergues, 2007]
- cost-register automata [Alur, D'Antoni, Deshmukh, Raghothaman, Yuan 2013]
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VASS models   Transducers   Pushdown automata

But all three papers are interesting for different reasons

Nonlinear extensions of weighted automata

Cost-register automata
[Many people]

Weighted MSO
[Droste and Gastin, 2005]

Weighted CFG
[Baker, 1979]

Nonlinear extensions of weighted automata

Cost-register automata	Weighted MSO	Weighted CFG
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- Assume $\Sigma = \{a\}$, so $|\Sigma| = 1$ and all models recognise sequences

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A context-free grammar assigns the number of derivation trees for a^n.

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- For example: grammar G: $X \to a$, $X \to aXX$

$G(1) = 1$ and $G_n = \sum_{i=1}^{n-2} G_i G_{n-i-1} \rightsquigarrow$ Catalan numbers (shifted)

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Corollary

Cost-register automata do not contain Weighted CFG and Weighted MSO

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- Properties: asymptotic bounds, periodicity and cancelling polynomials
- We proved that Catalan numbers C_n and n^n are not polynomial recursive.
- Are cost-register automata included in weighted MSO?

  F_{F_n} are polynomial recursive (F_n Fibonacci)

  Conjecture: F_{F_n} are not in Weighted MSO