When are Emptiness and Containment Decidable for Probabilistic Automata?

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Automata

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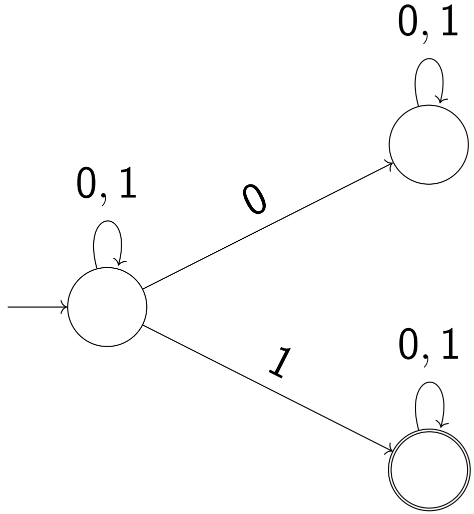
$\mathcal{A}$



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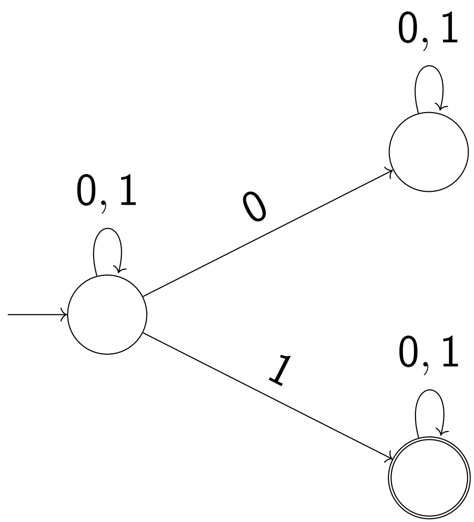

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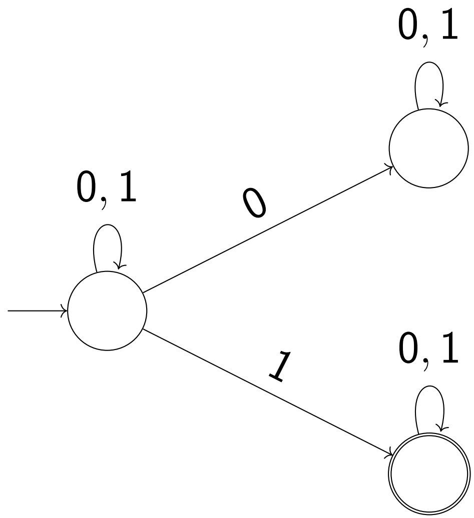

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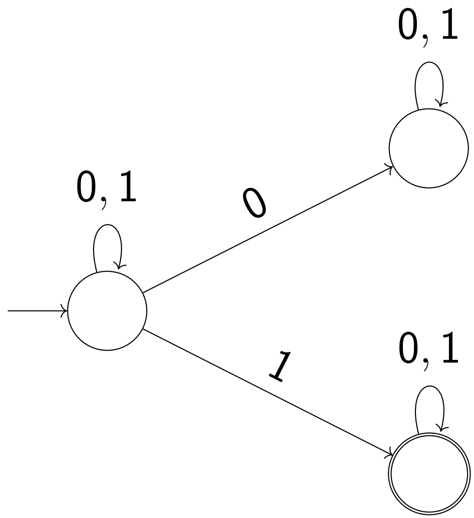

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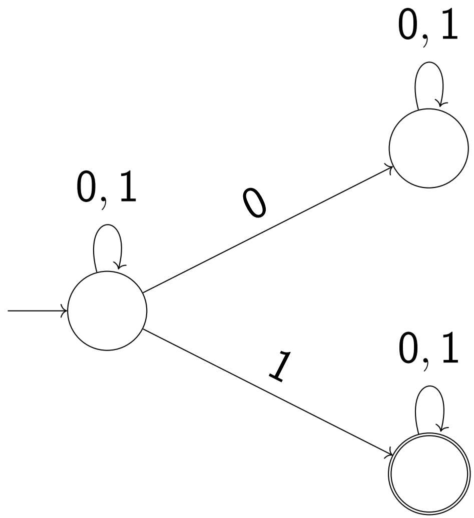

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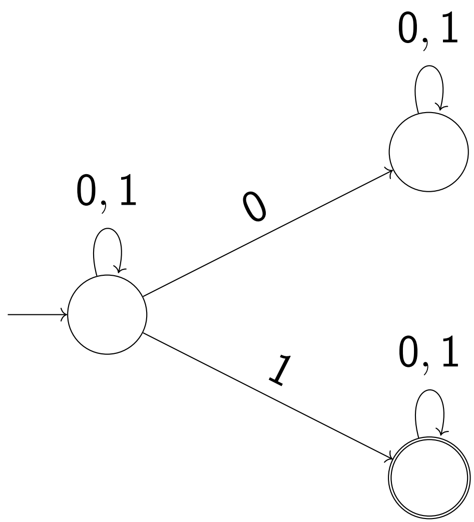

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artificial intelligence, verification of probabilistic systems...

Probabilistic automata (PA)

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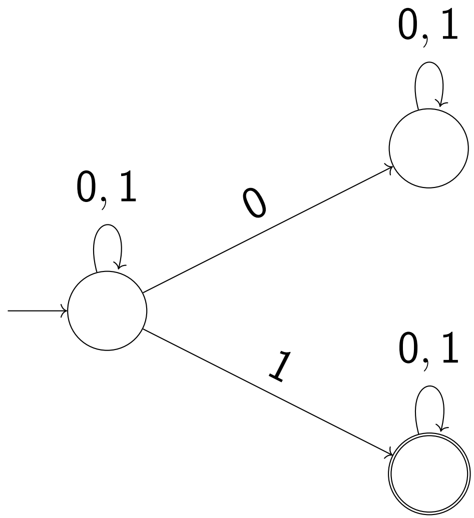

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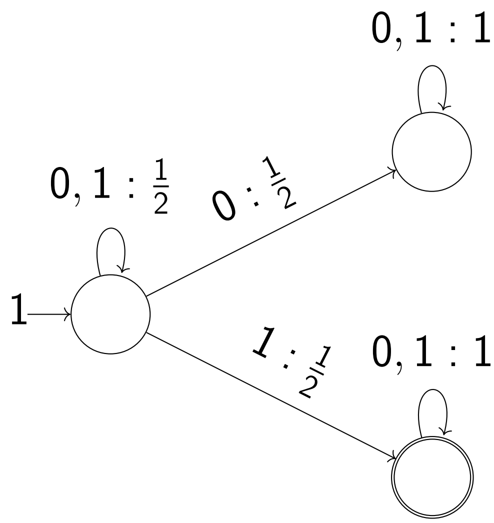

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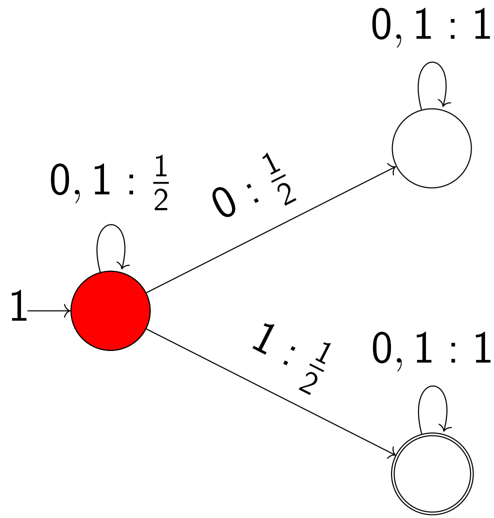

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transitions induce a distribution $(\frac{1}{2}, \frac{1}{2})$

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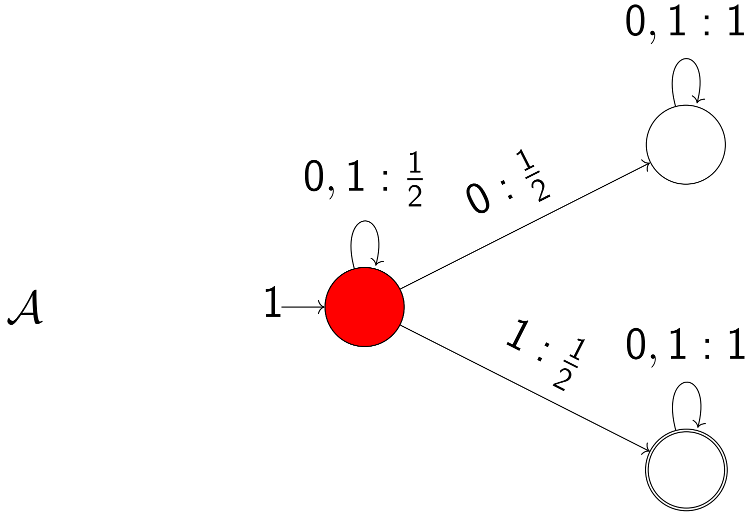

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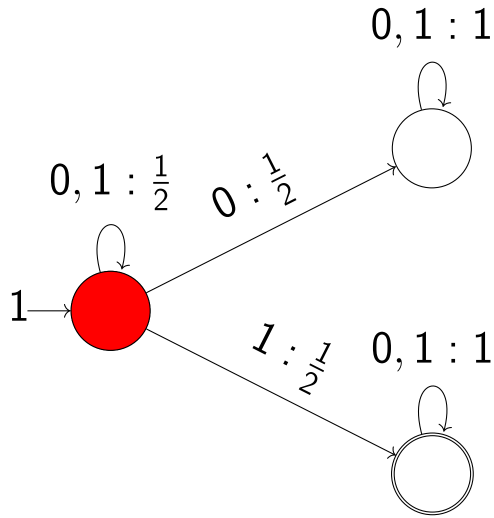

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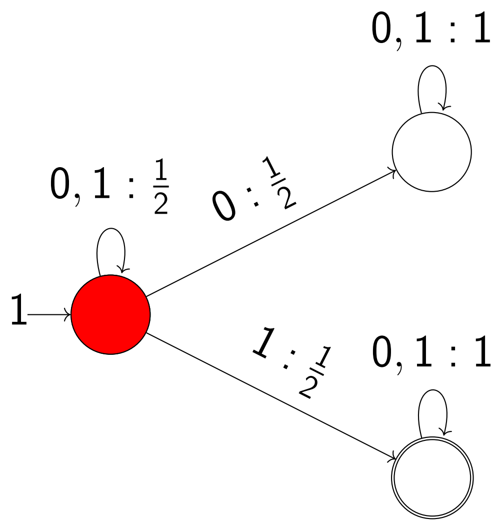

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E.g. $[\![\mathcal{A}]\!](1010) = 2^{-1} + 2^{-3}, \quad [\![\mathcal{A}]\!](w) = \mathrm{bin}(0.w) = \sum_{i | w(i)=1} 2^{-i}$

Decision problems

What do we study?

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<u>Emptiness problem</u>

$\textsc{Given}$: a PA automaton $\mathcal{A}$

$\textsc{Decide}$: does $[\![\mathcal{A}]\!](w) \leq \frac{1}{2}$ hold for all w?

($[\![\mathcal{A}]\!] \leq \frac{1}{2}$ in short)

Decision problems

What do we study?

ϵ-gap emptiness problem

GIVEN: ϵ, a PA automaton $\mathcal{A}$ s.t. either $[\![\mathcal{A}]\!] \leq \frac{1}{2}$ or $[\![\mathcal{A}]\!](w) > \frac{1}{2} + \epsilon$

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Remark

Emptiness reduces to containment (define $[\![\mathcal{B}]\!] = \frac{1}{2}$)

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tl;dr: it's all almost always undecidable

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- But equivalence is in PTIME [Schützenberger, 1961]
- Many subclasses with some decidability results:
  hierarchical, leaktight, bounded ambiguity

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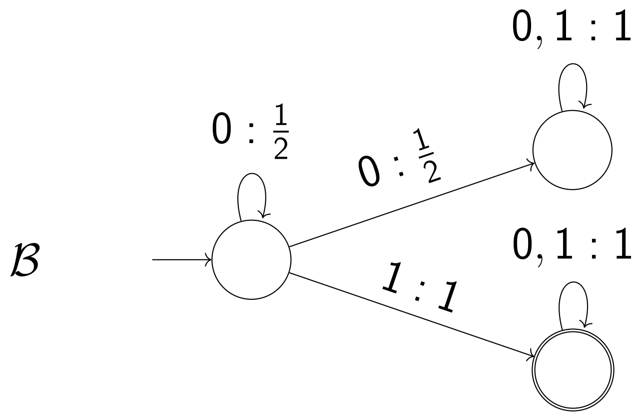

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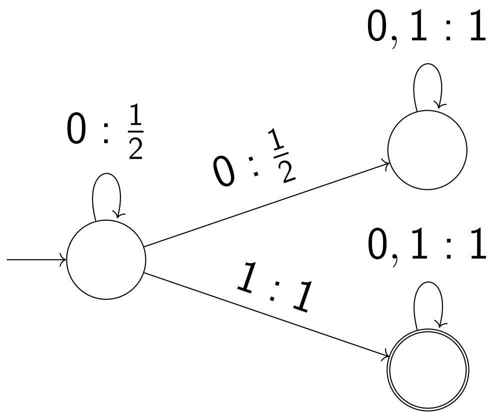

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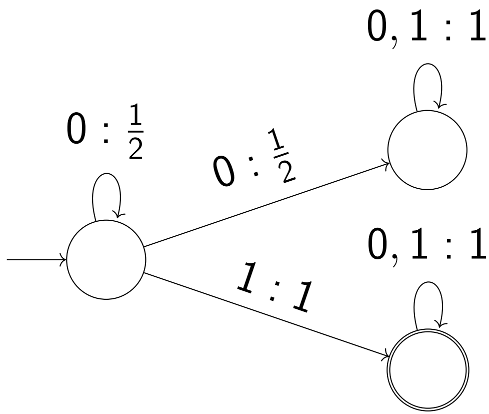

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Remark

Nothing between finitely ambiguous and linearly ambiguous PA

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(only in the submitted journal version)

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rest of the talk

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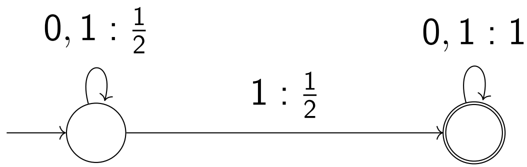

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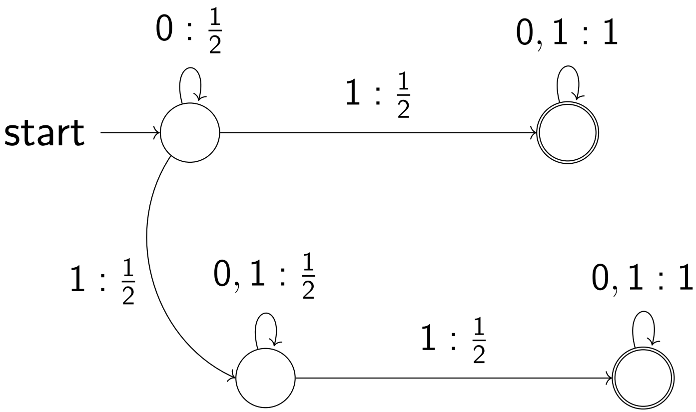

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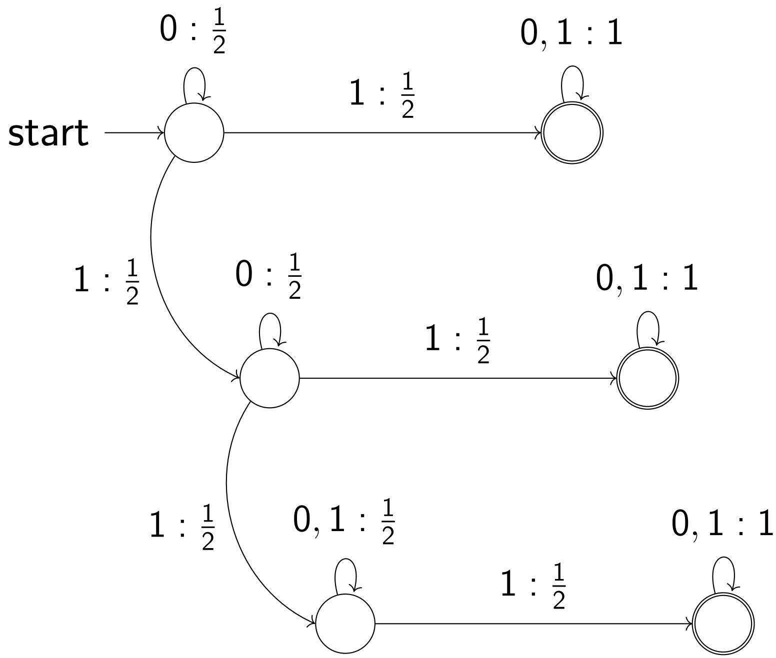

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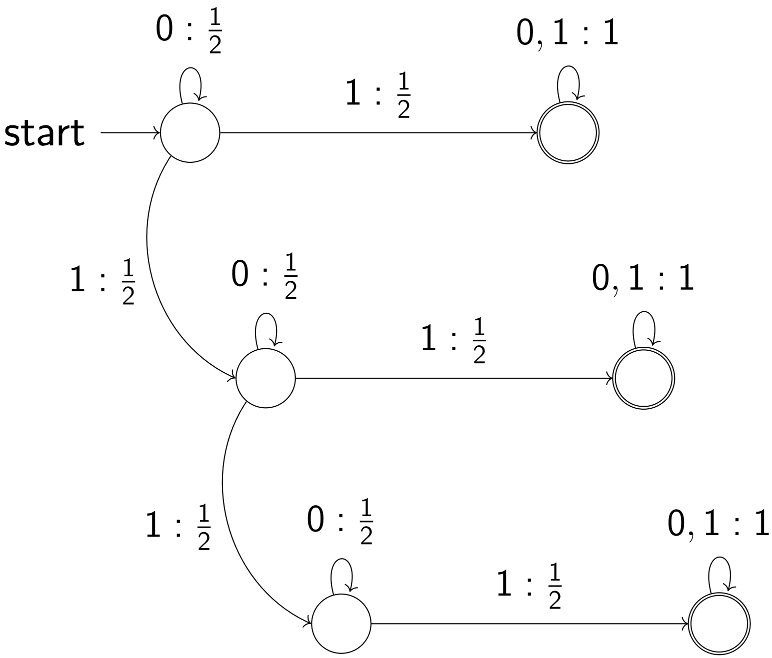

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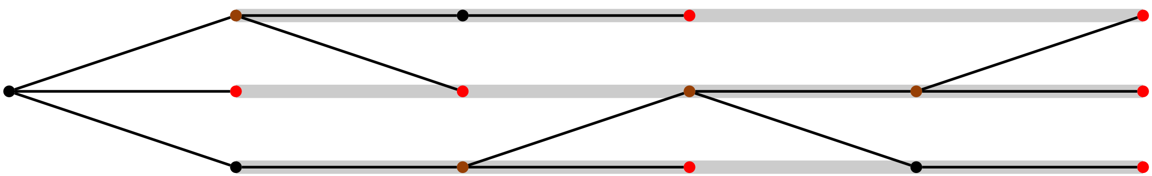
Lemma

Consider a rooted tree T of fixed width.

Suppose that the number of split nodes is at most m on every branch,

then the number of leaves in T is polynomial in m



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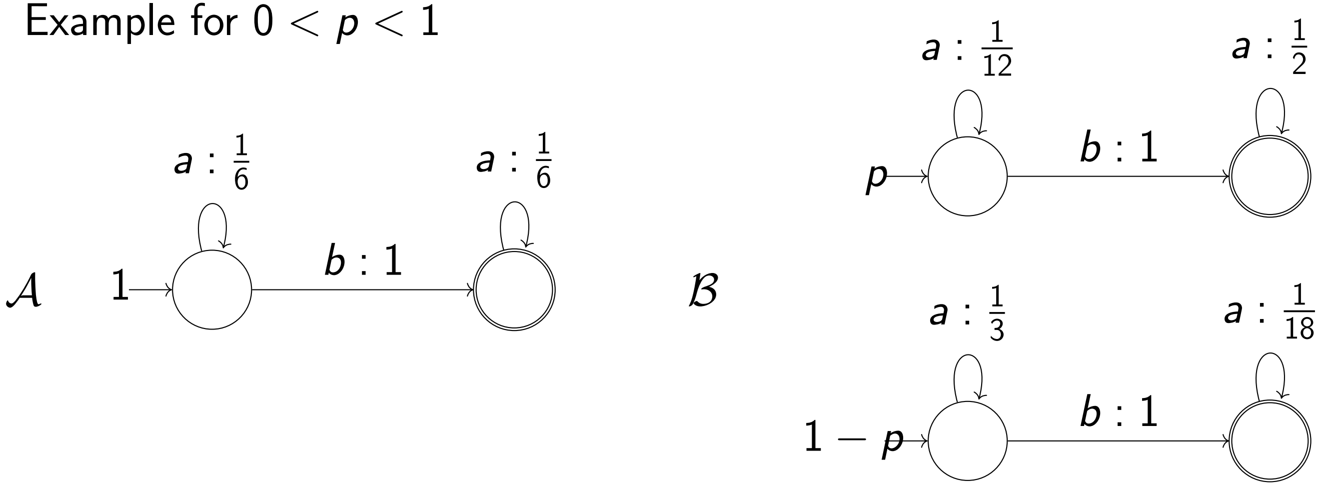

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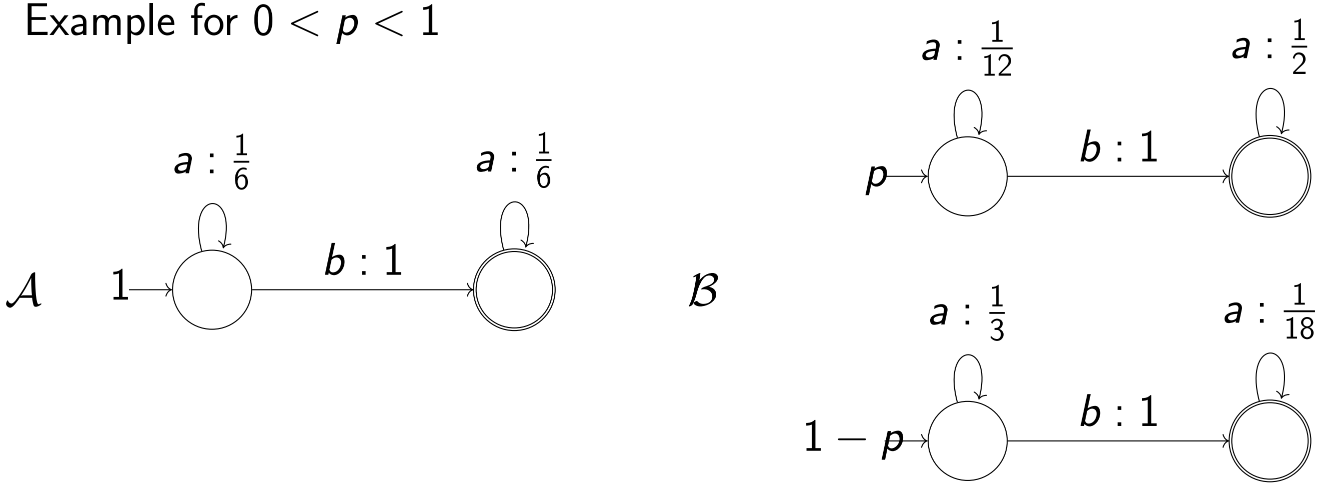

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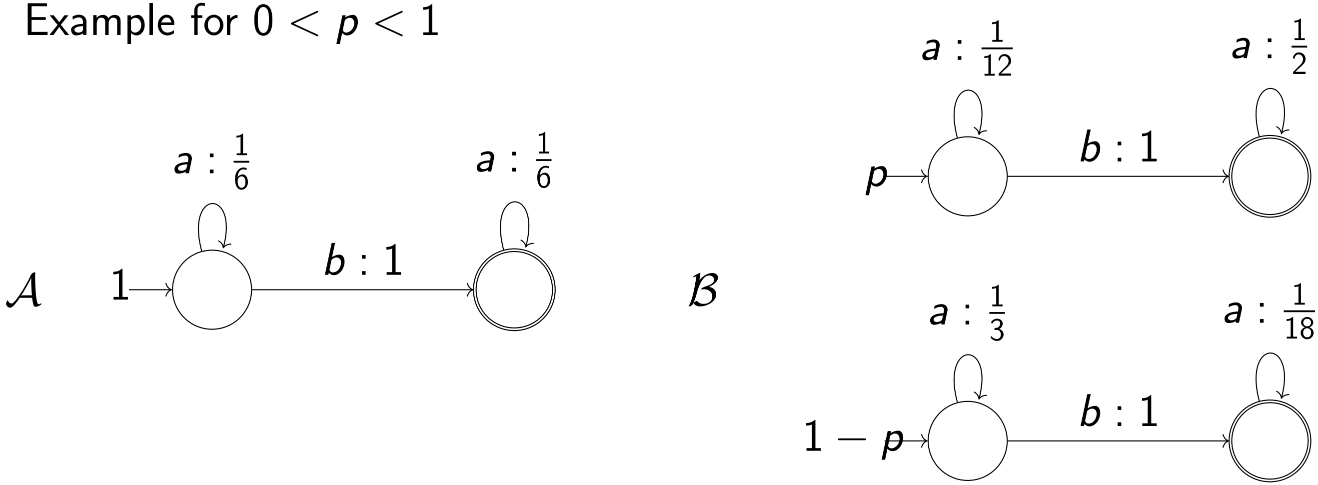

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- Suffices to solve: does for all $x_1, x_2 \in \mathbb{N}$ hold

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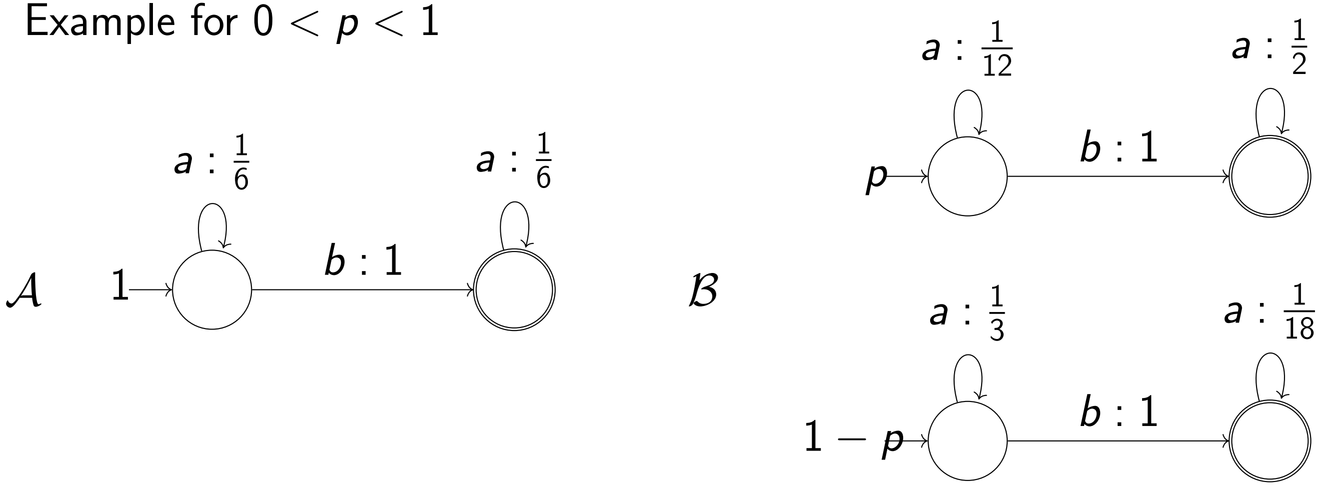

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$$1 \leq p\left(\tfrac{1}{2}\right)^{x_1}(3)^{x_2} + (1-p)(2)^{x_1}\left(\tfrac{1}{3}\right)^{x_2}$$

Reformulating containment of finitely ambiguous PA

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Exponential inequalities

GIVEN: $k, l, n > 0$, and vectors $\boldsymbol{p} \in \mathbb{Q}_{>0}^{k}$ and $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_k \in \mathbb{Q}_{>0}^{n}$

$$\boldsymbol{r} \in \mathbb{Q}_{>0}^{l} \text{ and } \boldsymbol{s}_1, \ldots, \boldsymbol{s}_l \in \mathbb{Q}_{>0}^{n}$$

DECIDE: for every $x_1, \ldots, x_n \in \mathbb{N}$ does it hold

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Reformulating containment of finitely ambiguous PA

$\mathcal{A} \leq \mathcal{B}$?

Exponential inequalities

GIVEN: $k, l, n > 0$, and vectors $\boldsymbol{p} \in \mathbb{Q}^k_{>0}$ and $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_k \in \mathbb{Q}^n_{>0}$

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Proposition

Containment of finitely ambiguous PA and exponential inequalities are effectively equi-decidable

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decidable
(without
Shanuel)

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$$\begin{aligned} &\exp(\log(p) - x_1 \log(2) + x_2 \log(3)) \\ + &\exp(\log(1 - p) + x_1 \log(2) - x_2 \log(3)) \end{aligned} \quad < \; 1$$

Geometric presentation

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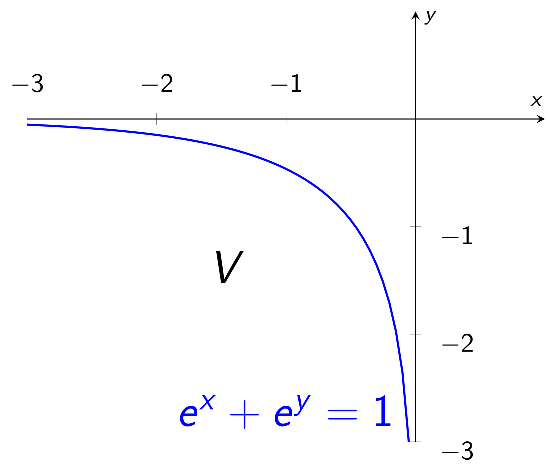
$$V = \{(x,y) \in \mathbb{R}^2 \mid e^x + e^y < 1\} \quad (\log = \log_e)$$



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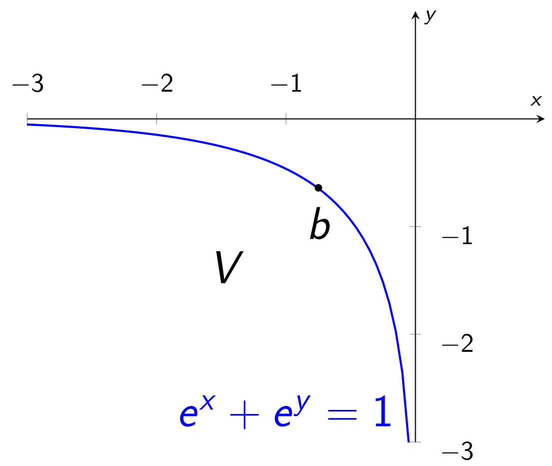

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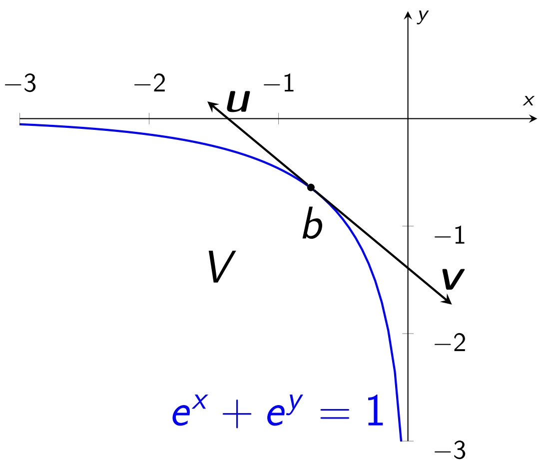

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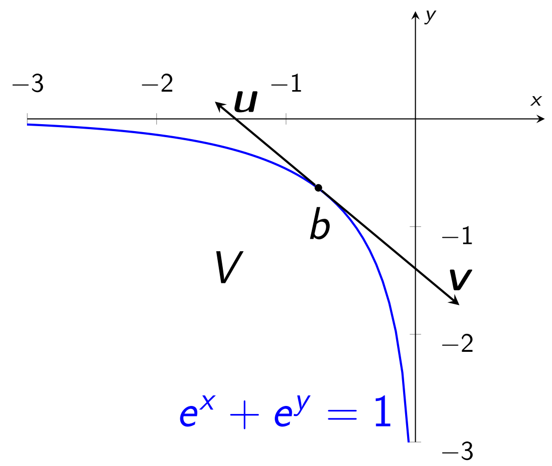

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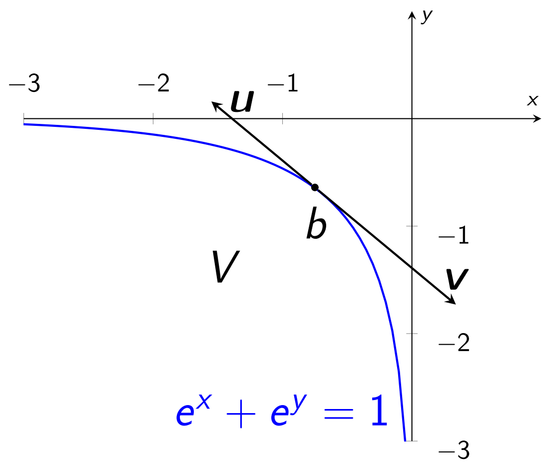

Back to the problem

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Corollary

(1) + (2) give decidability

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If $X \cap \mathbb{Z}^n = \emptyset$ then exist $\boldsymbol{d} \in \mathbb{Z}^n$, $a, b \in \mathbb{Z}$ s. t. $\{\boldsymbol{d}^\top \boldsymbol{x} \mid \boldsymbol{x} \in X\} \subseteq [a, b]$

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Satisfiability decidable assuming Schanuel [Macintyre, Wilkie; 1996]

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Previous lemma guarantees semi-decidability ($\mathbb{Z}$ to $\mathbb{N}$ is a technicality)

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- We solved it assuming:

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$V' = \{(x_1, \ldots, x_k, y_1, \ldots, y_l) \mid e^{x_1} + \ldots + e^{x_k} < e^{y_1} + \ldots + e^{y_l}\}$???