The Reachability Problem for Petri Nets is Not Elementary

Wojciech Czerwiński[1], Sławomir Lasota[1], Ranko Lazić[2],
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STOC 2019

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Petri net (d, T): d – dimension, $T \subseteq \mathbb{Z}^d$

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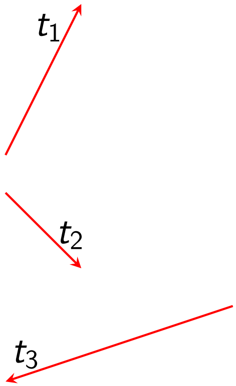

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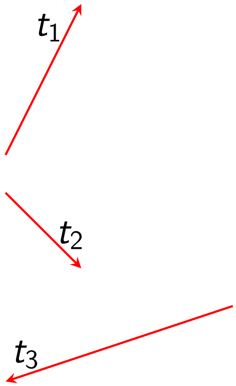

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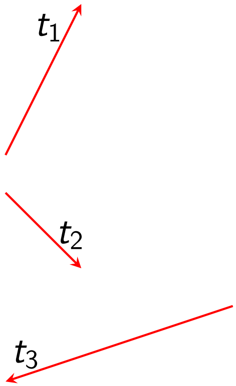

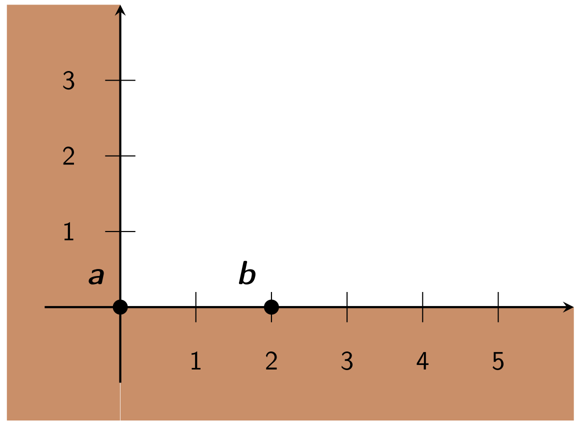

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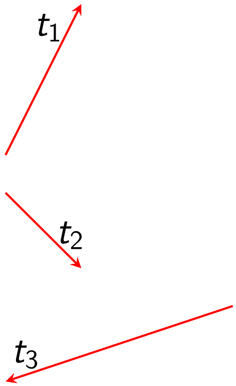

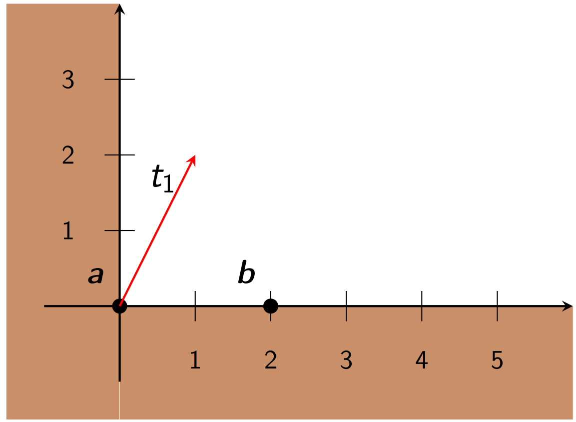

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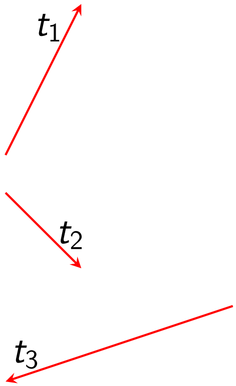

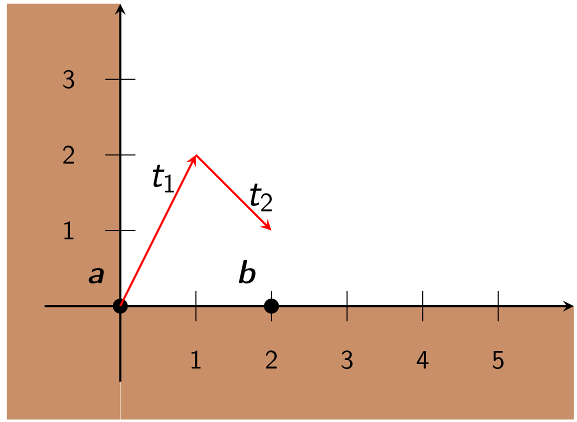

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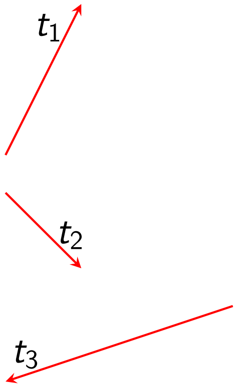

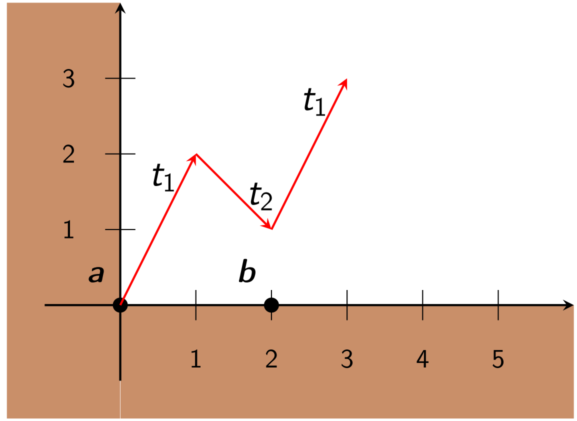

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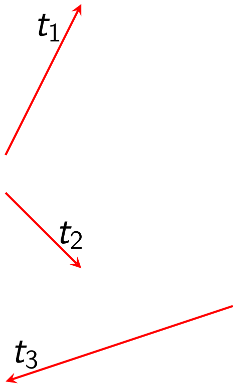

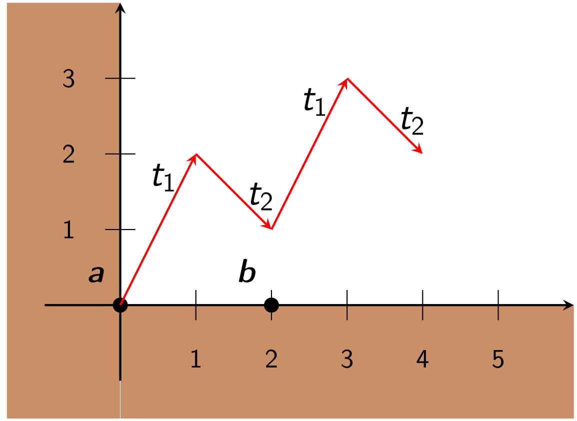

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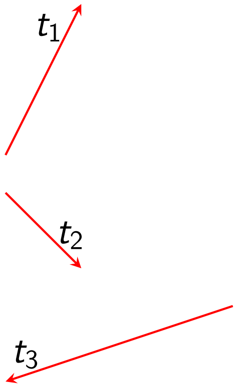

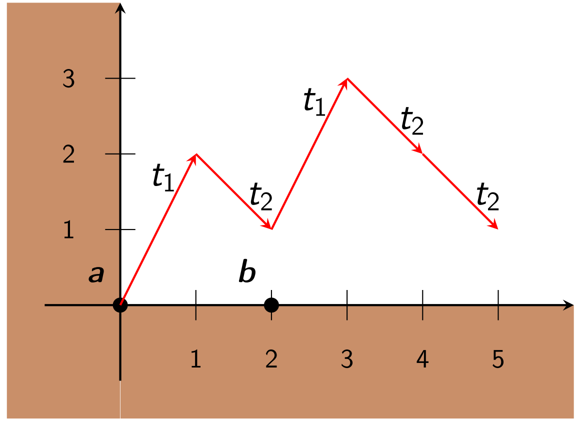

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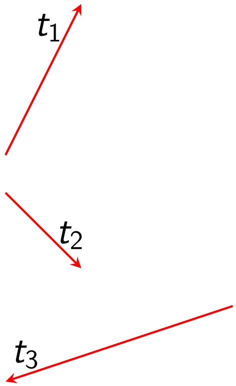

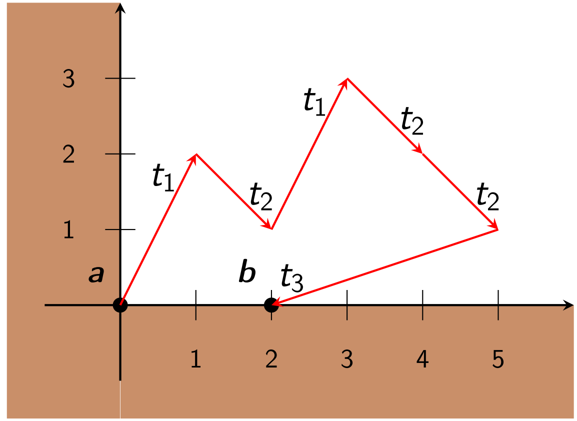

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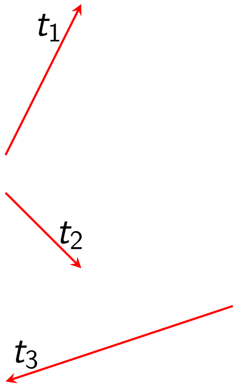

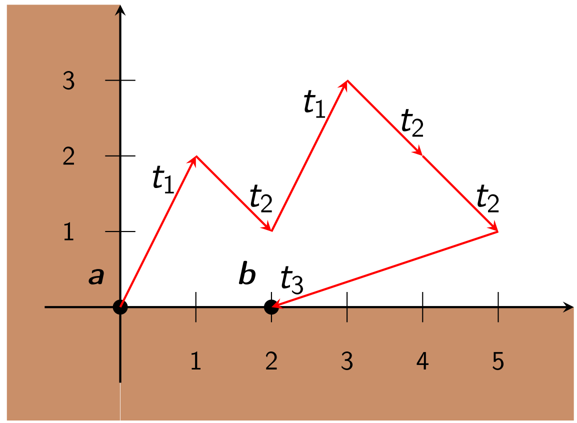

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One of the fundamental models and problems in formal verification

Verification of concurrent programs

1: x = True
2: **if** x **then goto** 3 **else goto** 1
3: x = False
4: # critical section
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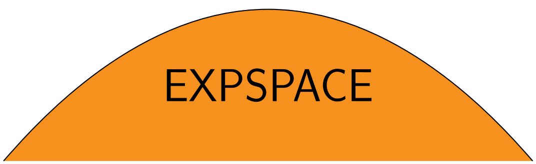
- Given two configurations $\boldsymbol{a} = (n, 0, 0, 0, 0)$ and $\boldsymbol{b} = (0, 0, 0, 2, 0)$

$\boldsymbol{b}$ is reachable from $\boldsymbol{a}$ $\iff$ two processes at once in critical section

Reachability for Petri nets state of art

What is the complexity?

1976 EXPSPACE-hard (Lipton)



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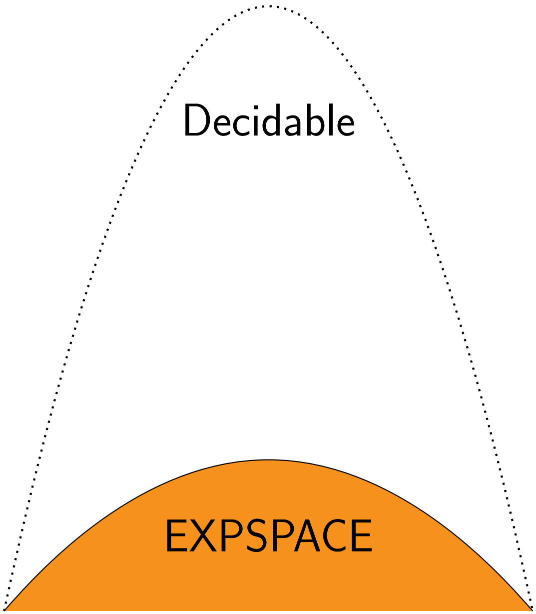

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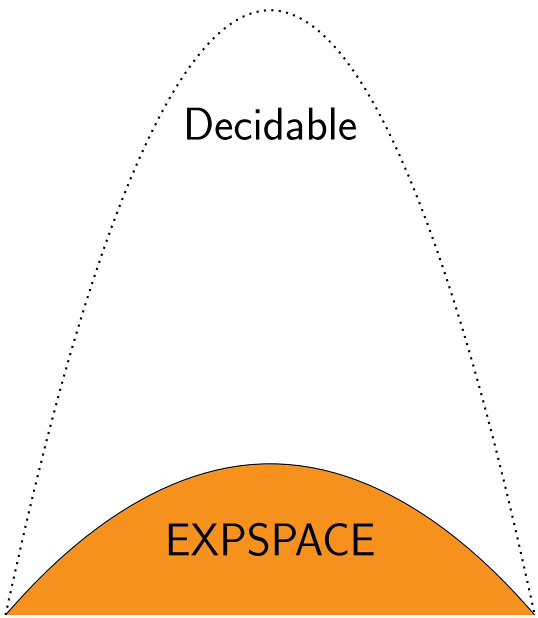

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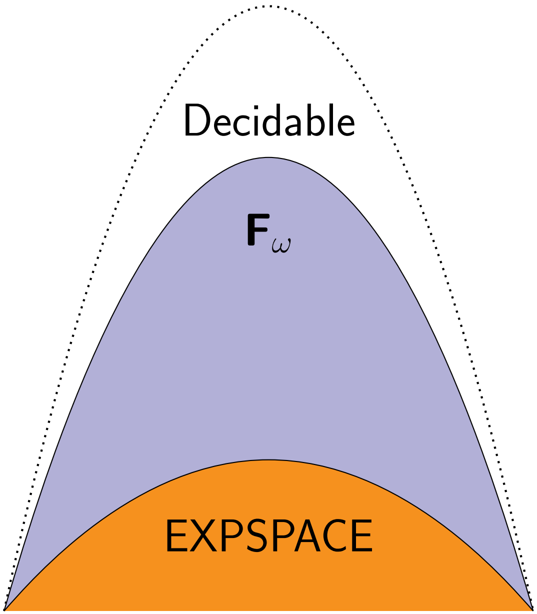

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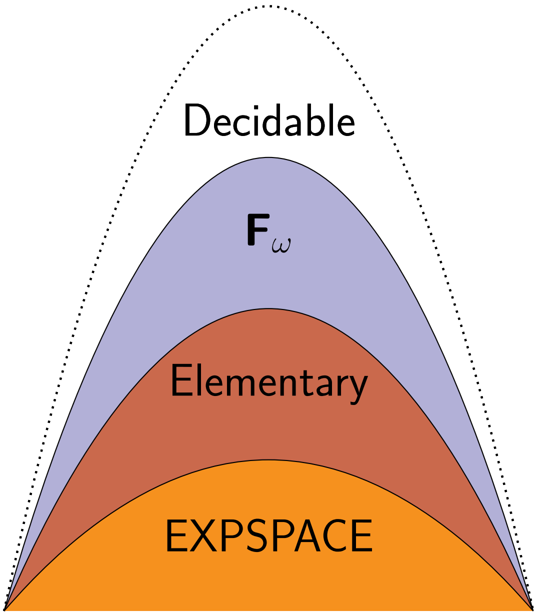
2015	In $\mathbf{F}_{\omega^3}$ (Leroux and Schmitz)
2019	In $\mathbf{F}_\omega$ (Leroux and Schmitz)

Reachability for Petri nets state of art

What is the complexity?

the first improvement of the lower bound



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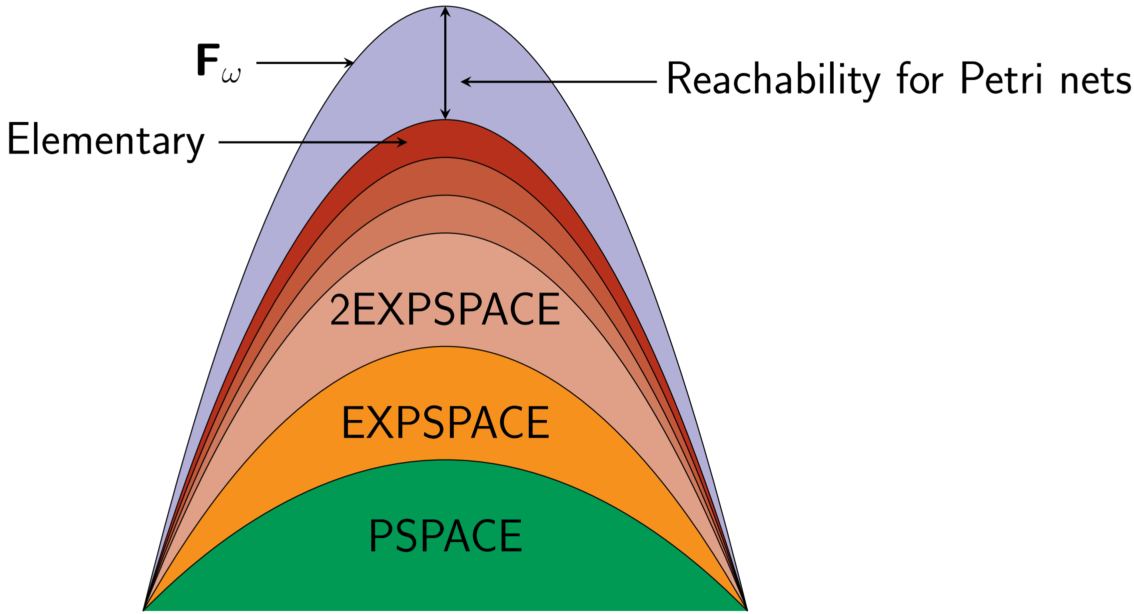
2019	In $\mathbf{F}_\omega$ (Leroux and Schmitz)
2019	Not elementary (this result)

Implications of the Not elementary result

- Disproves the EXPSPACE-complete conjecture

- Many problems are now not elementary, in: logic, databases, ...



Petri nets as counter programs

$\mathrm{x} \mathrel{+}= \mathrm{m}$	(add m to variable x)
$\mathrm{x} \mathrel{-}= \mathrm{m}$	(subtract m from variable x)
halt	(terminate)
halt if $\mathrm{x}_1, \ldots, \mathrm{x}_l = 0$	(terminate if listed variables are zero)
loop	(to be explained)

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Example

$x_1 \mathrel{+}= 3$
loop
  $x_1 \mathrel{-}= 1 \quad x_2 \mathrel{+}= 2$
until $x_1 = 0$
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until $x_1 = 0$

halt.

x_1	0
x_2	0

Petri nets as counter programs

x += m	(add *m* to variable x)
x −= m	(subtract *m* from variable x)
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Example

→ x_1 += 3
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x_1	0	3
x_2	0	0

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x $+=$ m	(add m to variable x)
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x_1	0	3	2	1
x_2	0	0	2	4

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x_1	0	3	2	1	0
x_2	0	0	2	4	6

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until $x_1 = 0$
→ **halt**.

in the end $x_2 = 3 \cdot 2 = 6$
"double the value"

x_1	0	3	2	1	0
x_2	0	0	2	4	6

Petri nets as counter programs

What is our **loop**

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Previous example

$x_1 \mathrel{+}= 3$
loop
$\quad x_1 \mathrel{-}= 1 \quad x_2 \mathrel{+}= 2$
until $x_1 = 0$
halt.
$\{x_2 = 6\}$

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$\{x_2 \in \{0, 2, 4, 6\}\}$

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x_1 += 3
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$\{x_2 \in \{0, 2, 4, 6\}\}$

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$\{x_2 = 6\}$

loop creates many runs

loop
  x += 1
halt

$\{x \in \mathbb{N}\}$

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$\{x_2 \in \{0, 2, 4, 6\}\}$

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loop
  $x \mathrel{+}= 1$
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$\{x \in \mathbb{N}\}$

Is there a run executing **halt**? $\iff$ reachability for Petri nets

Programs with normal loops

Additional command: **test** $x = 0$

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- More powerful and more readable

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Previous example: $n \to 2n$

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halt.

$x_1 \mathrel{+}= n$
loop
  $x_1 \mathrel{-}= 1 \quad x_2 \mathrel{+}= 2$
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Programs with normal loops

Additional command: **test** $x = 0$

- More powerful and more readable

Previous example: $n \to 2n$

```
x1 += n
loop
    x1 -= 1    x2 += 2
until x1 = 0
halt.
```

```
x1 += n
loop
    x1 -= 1    x2 += 2
halt if x1 = 0.
```

- But too powerful: reachability becomes undecidable

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- But too powerful: reachability becomes undecidable

How much can Petri nets simulate?

High level ideas to prove hardness

Key ingredient: compute a big value in some counter

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by iteratively reusing some program

- Previous program "double the value"

  take $\mathrm{x}_1, \ldots, \mathrm{x}_n$

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take $x_1, \ldots, x_n$

$$
\begin{array}{l}
x_1 \mathrel{+}= 2 \\
\textbf{loop} \\
\quad x_1 \mathrel{-}= 1 \quad x_2 \mathrel{+}= 2 \\
\textbf{loop} \\
\quad x_2 \mathrel{-}= 1 \quad x_3 \mathrel{+}= 2 \\
\vdots \\
\textbf{loop} \\
\quad x_{n-1} \mathrel{-}= 1 \quad x_n \mathrel{+}= 2 \\
\textbf{halt if } x_1, x_2, \ldots, x_{n-1} = 0
\end{array}
$$

High level ideas to prove hardness

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take $x_1, \ldots, x_n$

$x_1 \mathrel{+}= 2$
loop
$\quad x_1 \mathrel{-}= 1 \quad x_2 \mathrel{+}= 2$
loop
$\quad x_2 \mathrel{-}= 1 \quad x_3 \mathrel{+}= 2$
$\vdots$
loop
$\quad x_{n-1} \mathrel{-}= 1 \quad x_n \mathrel{+}= 2$
halt if $x_1, x_2, \ldots, x_{n-1} = 0$

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$x_1 \mathrel{+}= 2$
loop
 $x_1 \mathrel{-}= 1 \quad x_2 \mathrel{+}= 2$
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 $x_2 \mathrel{-}= 1 \quad x_3 \mathrel{+}= 2$
$\vdots$
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 $x_{n-1} \mathrel{-}= 1 \quad x_n \mathrel{+}= 2$
halt if $x_1, x_2, \ldots, x_{n-1} = 0$

$x_{i+1} = 2 \cdot x_i, \quad \text{so } x_n = 2^n$

Showing hardness

What programs to iterate?

- Lipton's program is "squaring": $x_{i+1} = (x_i)^2$, so $x_n = 2^{2^n}$
  (this gives the EXPSPACE lower bound from 1976)

- We managed to do "factorial": $x_{i+1} = (x_i)!$, so $x_n \geq \left. 2^{\cdot^{\cdot^{\cdot^{2}}}} \right\} n$ times.
  (this gives the not elementary lower bound in our paper)

Key idea

For the program $k \rightarrow k!$

Key idea

For the program $k \to k!$

$\mathrm{i} \mathrel{+}= k$

loop

  $\mathrm{x} \mathrel{+}= 1 \quad \mathrm{y} \mathrel{+}= 1 \quad \mathrm{z} \mathrel{+}= 1$

loop

  loop

    $\mathrm{x} \mathrel{-}= \mathrm{i} \quad \mathrm{x}' \mathrel{+}= \mathrm{i} + 1$

  loop

    $\mathrm{x}' \mathrel{-}= 1 \quad \mathrm{x} \mathrel{+}= 1$

  $\mathrm{i} \mathrel{-}= 1$

loop

  $\mathrm{x} \mathrel{-}= (k+1) \quad \mathrm{y} \mathrel{-}= 1$

halt if $\mathrm{y}, \mathrm{i} = 0$

Key idea

For the program $k \to k!$

$\mathrm{i} \mathrel{+}= k$
loop
  $\mathrm{x} \mathrel{+}= 1 \quad \mathrm{y} \mathrel{+}= 1 \quad \boxed{\mathrm{z} \mathrel{+}= 1}$
loop
  loop
    $\mathrm{x} \mathrel{-}= \mathrm{i} \quad \mathrm{x}' \mathrel{+}= \mathrm{i} + 1$
  loop
    $\mathrm{x}' \mathrel{-}= 1 \quad \mathrm{x} \mathrel{+}= 1$
  $\mathrm{i} \mathrel{-}= 1$
loop
  $\mathrm{x} \mathrel{-}= (k+1) \quad \mathrm{y} \mathrel{-}= 1$
halt if $\mathrm{y}, \mathrm{i} = 0$

$\boxed{\{z = k!\}}$

Key idea

For the program $k \to k!$

$\mathrm{i} \mathrel{+}= k$ i goes from k to 0 in the main loop

```
loop
    x += 1    y += 1    z += 1
loop
    loop
        x −= i    x′ += i + 1
    loop
        x′ −= 1    x += 1
    i −= 1
loop
    x −= (k + 1)    y −= 1
halt if y, i = 0
```

$\{z = k!\}$

Key idea

For the program $k \to k!$

$\text{i} \mathrel{+}= k$

loop

$\text{x} \mathrel{+}= 1 \quad \text{y} \mathrel{+}= 1 \quad \text{z} \mathrel{+}= 1$

loop

loop

$\text{x} \mathrel{-}= \text{i} \quad \text{x}' \mathrel{+}= \text{i} + 1$

loop

$\text{x}' \mathrel{-}= 1 \quad \text{x} \mathrel{+}= 1$

$\text{i} \mathrel{-}= 1$

loop

$\text{x} \mathrel{-}= (k+1) \quad \text{y} \mathrel{-}= 1$

halt if $\text{y}, \text{i} = 0$

$\{z = k!\}$

i goes from k to 0 in the main loop

$\text{x}, \text{y}, \text{z}$ initialised to any c

every time x multiplied by at most $\frac{\text{i}+1}{\text{i}}$

Key idea

For the program $k \to k!$

$\mathsf{i} \mathrel{+}= k$

loop

$\mathsf{x} \mathrel{+}= 1 \quad \mathsf{y} \mathrel{+}= 1 \quad \mathsf{z} \mathrel{+}= 1$

loop

loop

$\mathsf{x} \mathrel{-}= \mathsf{i} \quad \mathsf{x}' \mathrel{+}= \mathsf{i} + 1$

loop

$\mathsf{x}' \mathrel{-}= 1 \quad \mathsf{x} \mathrel{+}= 1$

$\mathsf{i} \mathrel{-}= 1$

loop

$\mathsf{x} \mathrel{-}= (k+1) \quad \mathsf{y} \mathrel{-}= 1$

halt if $\mathsf{y}, \mathsf{i} = 0$

$\{z = k!\}$

i goes from k to 0 in the main loop

x, y, z initialised to any c

every time x multiplied by at most $\frac{\mathsf{i}+1}{\mathsf{i}}$

In the end $\mathsf{x} \leq c \cdot \prod_{i=1}^{k} \frac{i+1}{i} = c \cdot (k+1)$

$\frac{k+1}{k} \cdot \frac{k}{k-1} \cdot \frac{k-1}{k-2} \cdots \frac{3}{2} \cdot \frac{2}{1} = k+1$

Key idea

For the program $k \to k!$

$\mathrm{i} \mathrel{+}= k$

loop

$\mathrm{x} \mathrel{+}= 1 \quad \mathrm{y} \mathrel{+}= 1 \quad \mathrm{z} \mathrel{+}= 1$

loop

loop

$\mathrm{x} \mathrel{-}= \mathrm{i} \quad \mathrm{x}' \mathrel{+}= \mathrm{i}+1$

loop

$\mathrm{x}' \mathrel{-}= 1 \quad \mathrm{x} \mathrel{+}= 1$

$\mathrm{i} \mathrel{-}= 1$

loop

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halt if $\mathrm{y}, \mathrm{i} = 0$

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But we need $\mathrm{x} \geq \mathrm{y} \cdot (k+1) = c \cdot (k+1)$

Key idea

For the program $k \to k!$

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loop
  $\mathrm{x} \mathrel{+}= 1$ $\mathrm{y} \mathrel{+}= 1$ $\mathrm{z} \mathrel{+}= 1$
loop
  loop
    $\mathrm{x} \mathrel{-}= \mathrm{i}$ $\mathrm{x}' \mathrel{+}= \mathrm{i}+1$
  loop
    $\mathrm{x}' \mathrel{-}= 1$ $\mathrm{x} \mathrel{+}= 1$
  $\mathrm{i} \mathrel{-}= 1$
loop
  $\mathrm{x} \mathrel{-}= (k+1)$ $\mathrm{y} \mathrel{-}= 1$
halt if $\mathrm{y}, \mathrm{i} = 0$
$\{z = k!\}$

i goes from k to 0 in the main loop

$\mathrm{x}, \mathrm{y}, \mathrm{z}$ initialised to any c

so $k \mid \mathrm{x}, \quad k-1 \mid \mathrm{x}, \ldots$

every time x multiplied by at most $\frac{\mathrm{i}+1}{\mathrm{i}}$

In the end $\mathrm{x} \leq c \cdot \prod_{i=1}^{k} \frac{i+1}{i} = c \cdot (k+1)$

But we need $\mathrm{x} \geq \mathrm{y} \cdot (k+1) = c \cdot (k+1)$

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- Several applications and corollaries
  - reachability is harder than we expected
  - many problems are not elementary (FO2 on data words)

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- This originated from studying 1-Pushdown-VASS

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  - reachability is harder than we expected
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So maybe it's good to study restrictions of generalizations of etc...

(my current favourite: BOBRVASS)