Timed pushdown automata and branching vector addition systems

Lorenzo Clemente[1], Sławomir Lasota[1], Ranko Lazić[2], and Filip Mazowiecki[3]

[1]University of Warsaw

[2]University of Warwick

[3]University of Oxford

LICS 2017
Reykjavik

Outline

1. Three models

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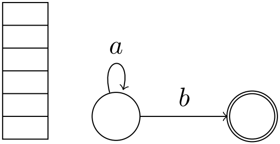
trPDA



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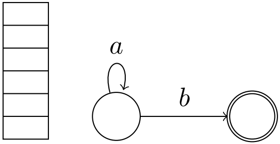

time registers x, y, z

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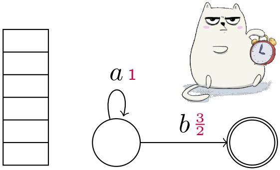

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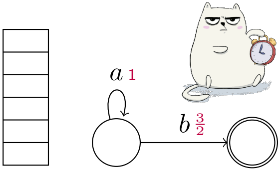

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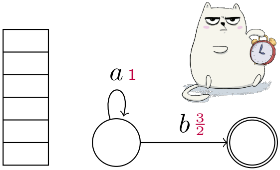

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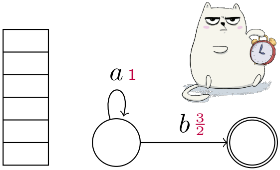

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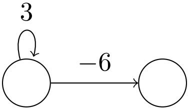

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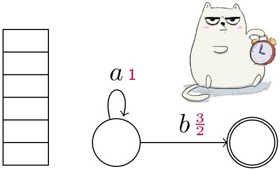

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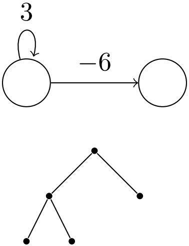

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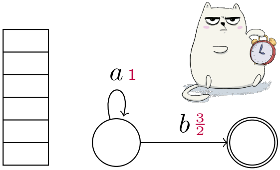

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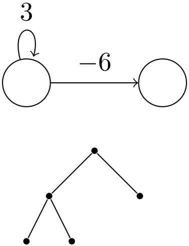

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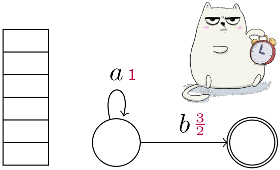

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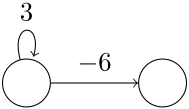

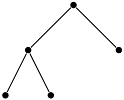

2. Reductions between models

3. Decidability

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What is time?

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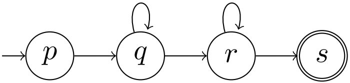

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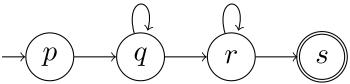

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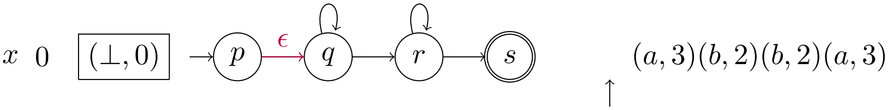

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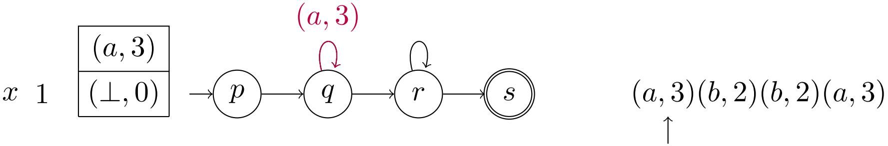

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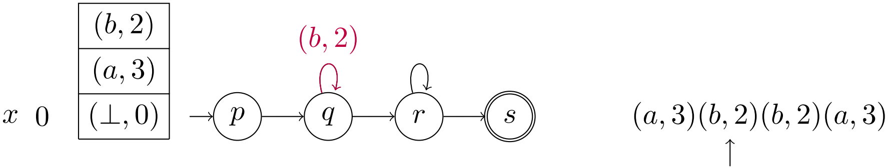

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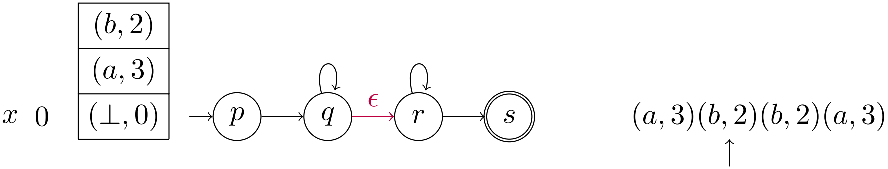

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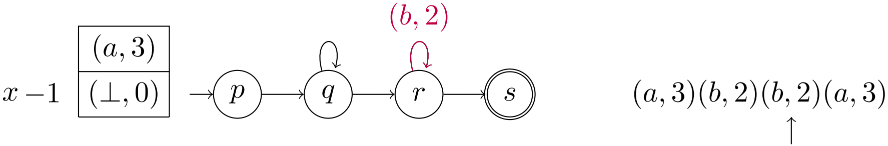

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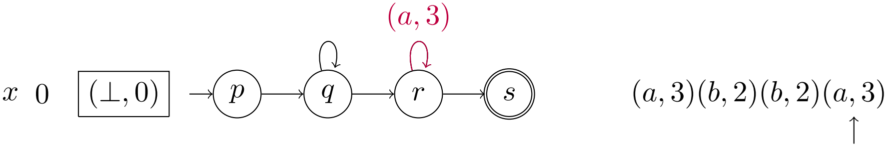

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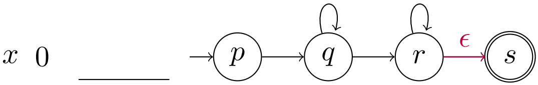

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$\uparrow$

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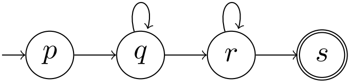

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- Non-monotonic time

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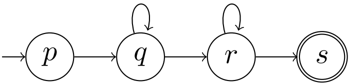

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- Non-monotonic time
- Only one register (or orbit-finiteness)

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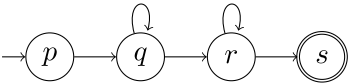

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Strictly subsumes other models:

- [Bouajjani, Echahed, Robbana]
- [Abdulla, Atig, Stenman]

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Problem: non-emptiness of $L(\mathcal{A})$

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Time stack – this paper

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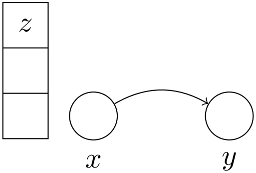

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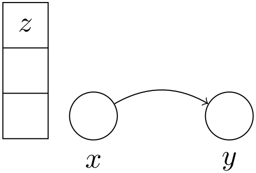

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$$(x = y + 1) \ \wedge \ (y \leq z + 1 + 1 + 1) \ \wedge \ (z \leq y + 1) \ \wedge \ (x \leq z)$$

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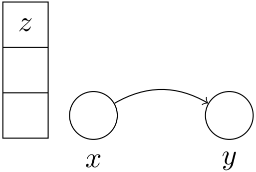

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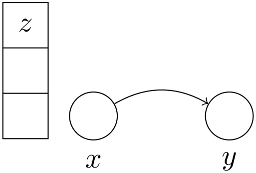

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Transition: 3 intervals

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Systems of equations $\mathcal{S}$ using: $\cup, \cap, +$ and $\{1\}, \{-1\}$

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Restricting $\cap$

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trPDA $\mathcal{A}$ with states Q, empty stack acceptance

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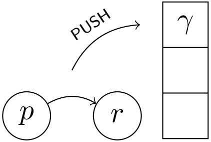

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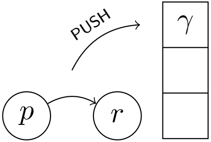

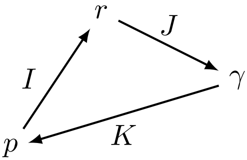

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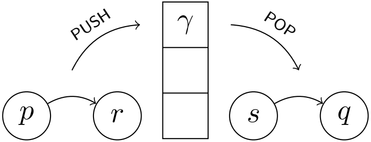

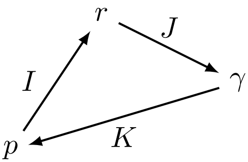

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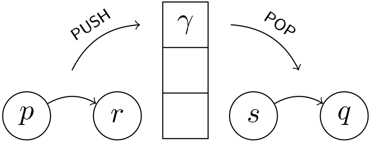

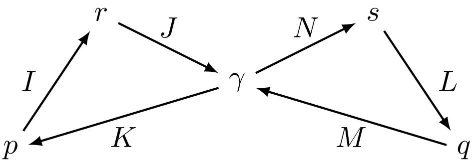

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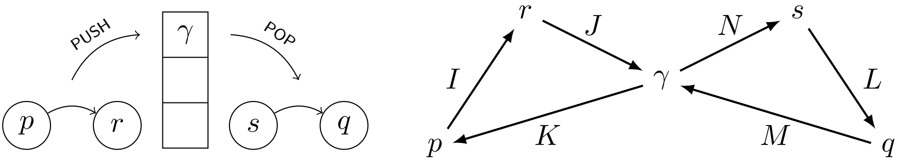

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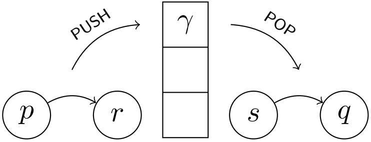

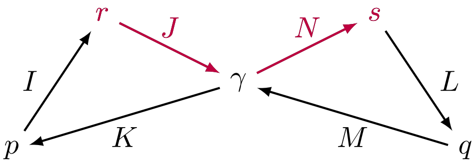

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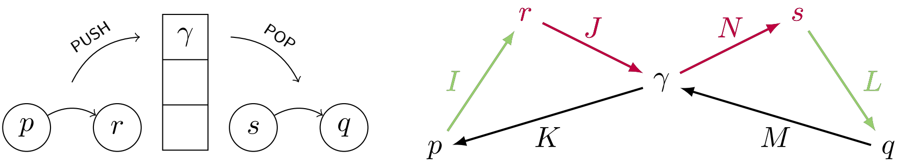

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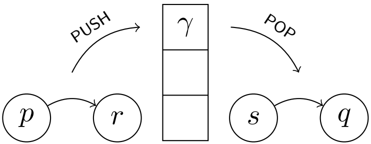

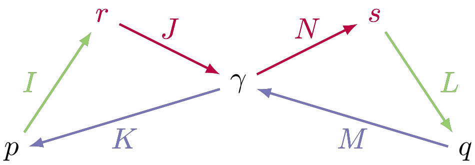

BVASS

Recall 1-VASS

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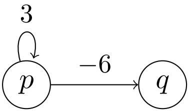

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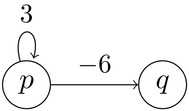

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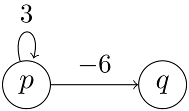

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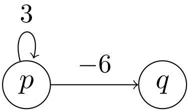

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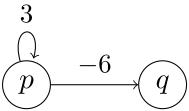

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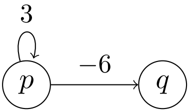

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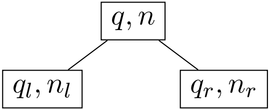

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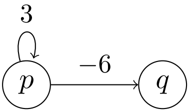

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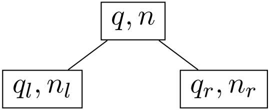

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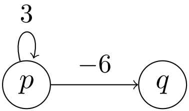

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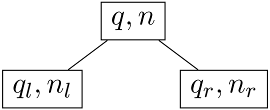

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In higher dimensions – undecidable ($d \geq 6$) [Lazić, 2010]

Decidability BVASS

$1\text{-BVASS}^{\pm}$ $\mathcal{B}$, configuration (q, n)

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If (q, n) is reachable then there is a computation with all values bounded by $N = poly(n) \cdot exp(|B|)$.

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So in $\textsc{ExpTime}$

Results summary

Three models/problems

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trPDA
(non-emptiness)

Results summary

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trPDA
(non-emptiness)

Systems
(non-emptiness)

Results summary

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trPDA (non-emptiness)	Systems (non-emptiness)	$1\text{-BVASS}^{\pm}$ (reachability)

Results summary

Three models/problems

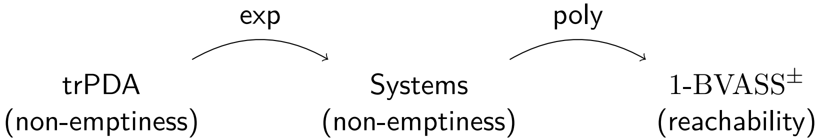
exp

trPDA (non-emptiness) → Systems (non-emptiness)

$1\text{-BVASS}^{\pm}$ (reachability)

Results summary

Three models/problems



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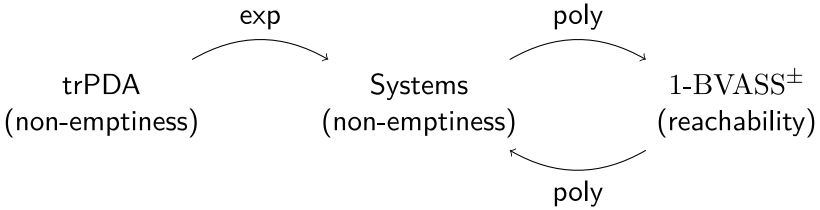

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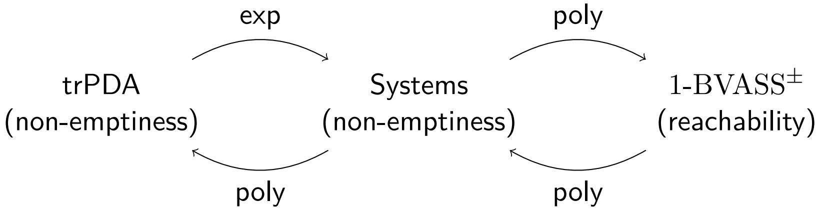

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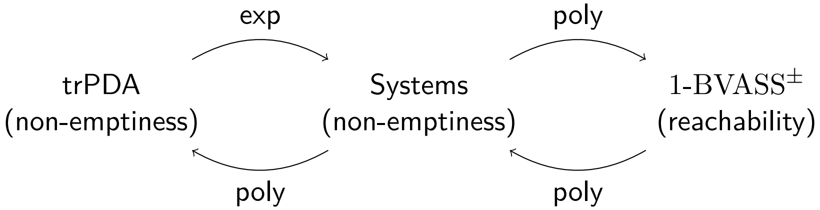

in EXPTIME

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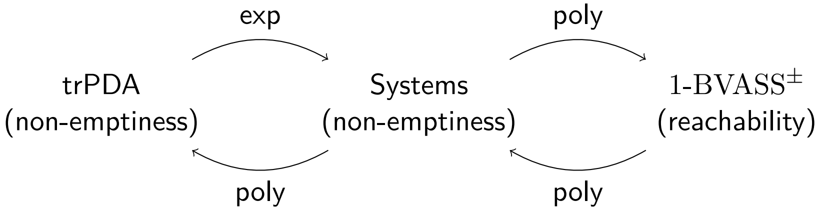

in 2EXPTIME	in EXPTIME	in EXPTIME

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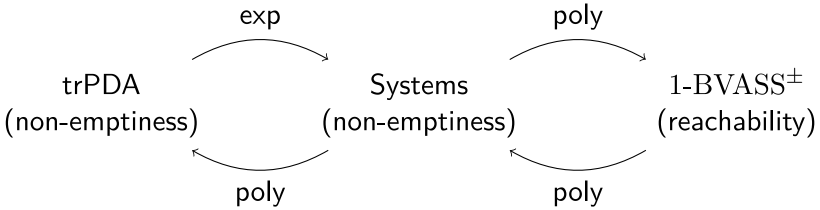

trPDA	Systems	1-BVASS$^{\pm}$
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EXPTIME-hard	PSPACE-hard	PSPACE-hard

Conclusions

- Complexity gaps

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- Reachability of $\mathrm{n\text{-}BVASS}^{\pm}$ for $n < 6$