Copyless cost-register automata

Filip Mazowiecki

Max Planck Institute for Software Systems, Germany

WATA 2020/2021

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Outline

1. Introduction: copyless linear CRA etc

2. Undecidability of equivalence

3. Future questions?

Weighted automata (WA)

Let $\mathbb{S}(\oplus, \odot)$ be a semiring, e.g. $\mathbb{Q}(+, \cdot)$ or $\mathbb{Z}(\min, +)$

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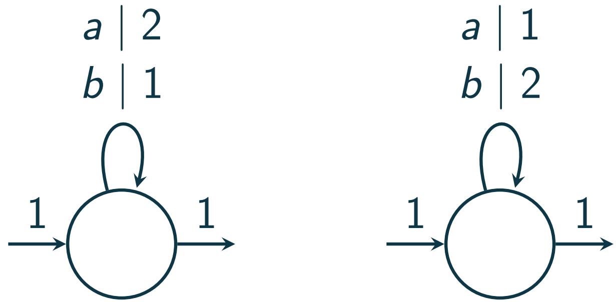

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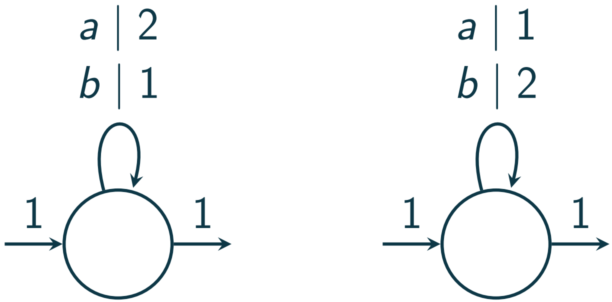

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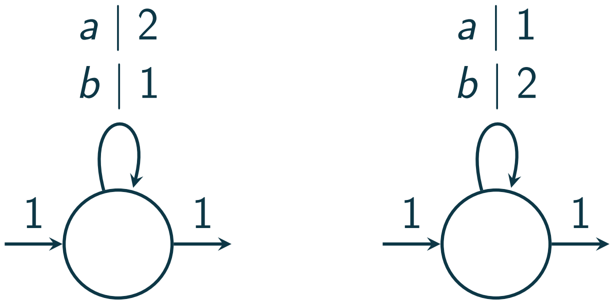

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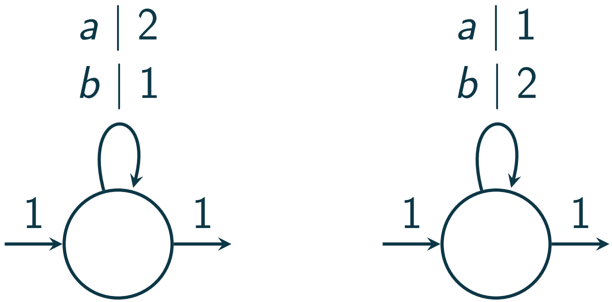

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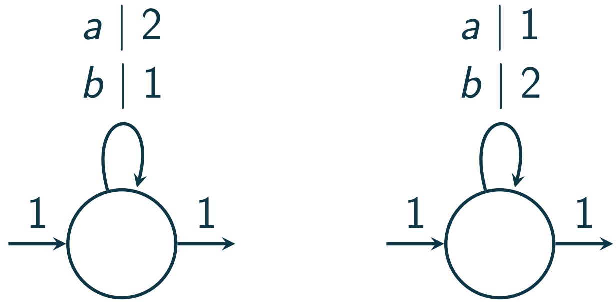

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$$\mathcal{A}(aabba) = I^\top M_a M_a M_b M_b M_a F$$

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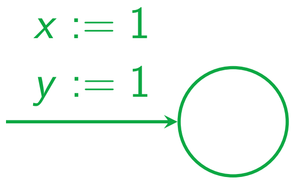

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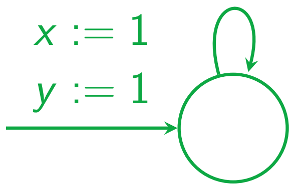

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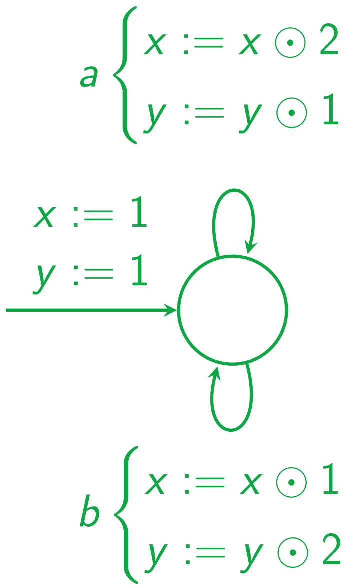

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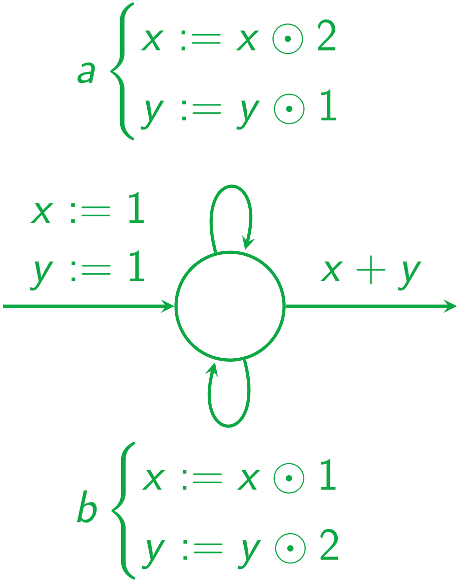

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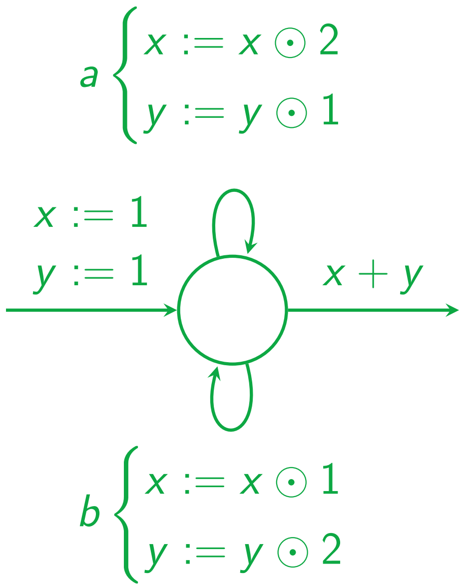

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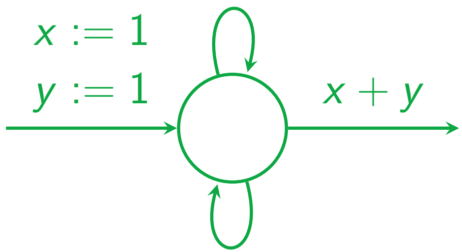

$$b \begin{cases} x := x \odot 1 \\ y := y \odot 2 \end{cases}$$

These are (almost) CRA

- States, deterministic transitions

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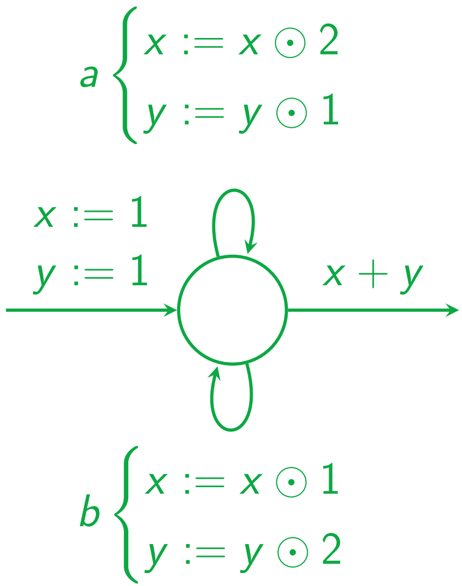

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- Linear CRA = WA (CRA are nonlinear in general)

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Notation:

- $A \subseteq B$: for all (commutative) semirings A is contained in B
- $A \not\subseteq B$: there exists a (commutative) semiring s.t. A is not contained in B

Subclasses compared

fin-amb WA

copyless linear CRA

$\subsetneq$

poly-amb WA

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poly-amb WA $\subsetneq$ WA

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A lot of my work is dedicated to understand this picture

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Recently in [Barloy et al., 2020] 1-letter WA over $\mathbb{Q}(+,\cdot)$

- WA = linear recursive sequences (LRS)
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mostly over $\mathbb{Z}(\min, +)$

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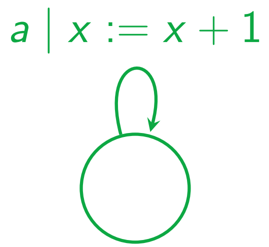
CRA



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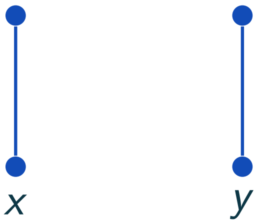

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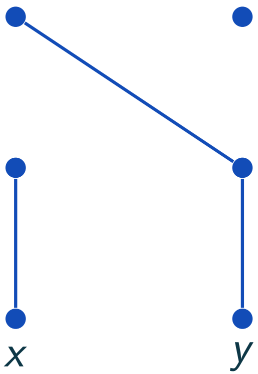

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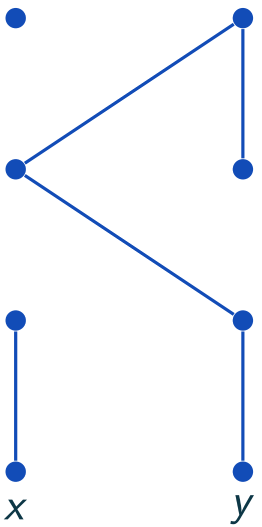

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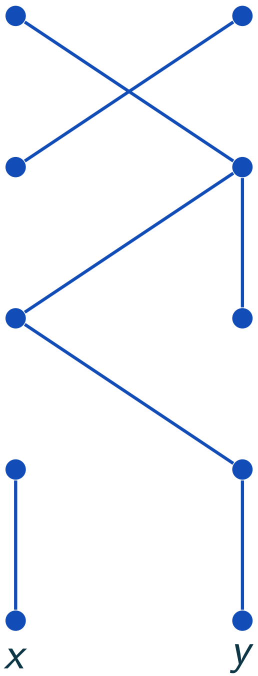

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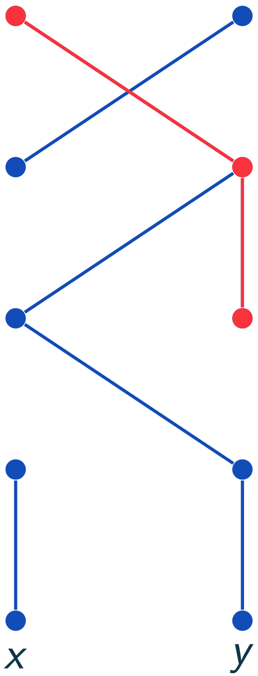

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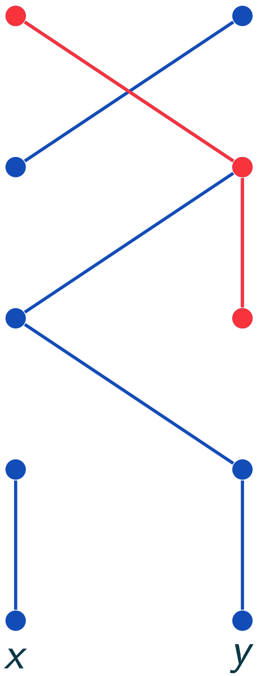

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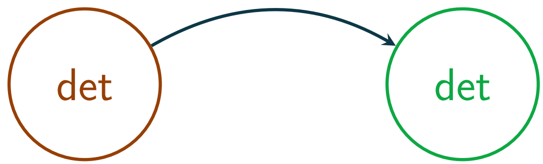

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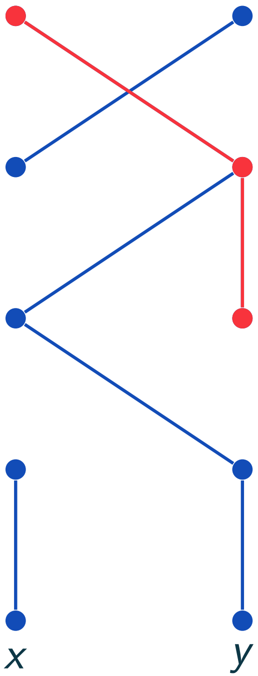

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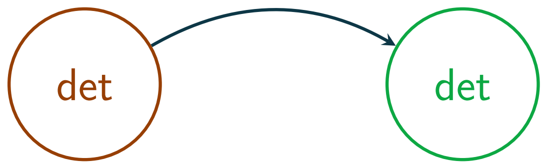

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Update of every register

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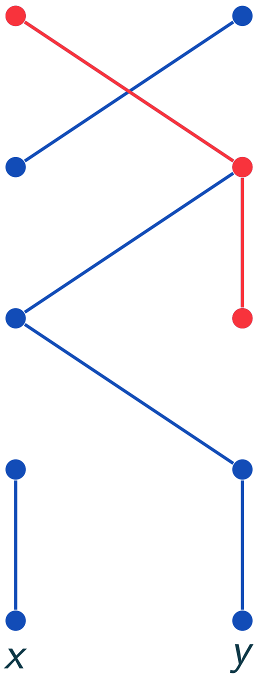

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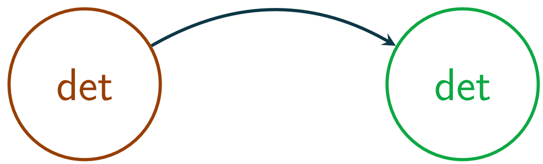

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This is even linear ambiguous

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Equivalence: given $\mathcal{A}$ and $\mathcal{B}$ is $\mathcal{A}(w) = \mathcal{B}(w)$ for all w?

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No :(and that's a big part of the talk

Decision problem

Equivalence: given $\mathcal{A}$ and $\mathcal{B}$ is $\mathcal{A}(w) = \mathcal{B}(w)$ for all w?

Over $\mathbb{Z}(\min, +)$ or $\mathbb{N}(\min, +)$

- Undecidable for polynomially ambiguous WA [Almagor et al., 2011]
- Decidable for finitely ambiguous WA [Filiot et al., 2017]

In both cases the proofs are standard

Is equivalence decidable for copyless linear CRA?

No :(and that's a big part of the talk

But we have new conjectures :)

Outline

1. Introduction: copyless linear CRA etc

2. Undecidability of equivalence

3. Future questions?

Undecidable equivalence

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The equivalence problem is undecidable for copyless linear CRA over $\mathbb{Z}(\min, +)$

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$\mathcal{A}$ is defined as above, $\mathcal{B} = 0$.

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Is equivalence still undecidable over $\mathbb{N}(\min, +)$?

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Naive fixes are not copyless

or not linear

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Ideally 0
should be $x^{\frac{u}{2}}$

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  x^{cb}, x^{2cb} are nonzero only when reading *cb*

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Lemma

Once registers are **dead** they always remain **dead**

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Wrapping up for more counters $x_1, \ldots, x_k$

If everything went ok then $x_i^0 = x_i^{\frac{u}{2}}$

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$e = 4k$
so all values always even

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- Output: in $\mathcal{A}$: $\min(x^{avg}, x_1^0 + 1, \dots, x_k^0 + 1)$, in $\mathcal{B}$: $\min(x_1^0 + 1, \dots, x_k^0 + 1)$

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  $\mathcal{A}$ outputs x^{avg} only if nothing was **dead** and x_i^0 are all equal

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2. Undecidability of equivalence

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Other fragments of weighted automata

Is there a nontrivial fragment of WA with decidable equivalence for $\mathbb{N}(\min, +)$?

Other fragments of weighted automata

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Maybe. Bounded alternation for CRA?

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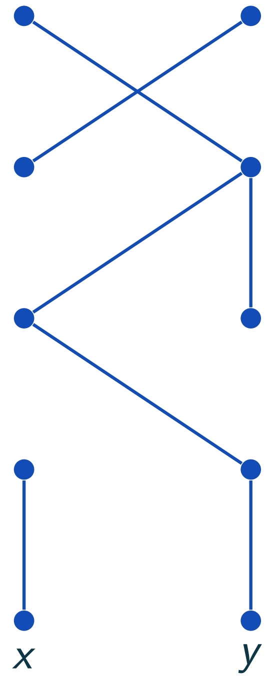

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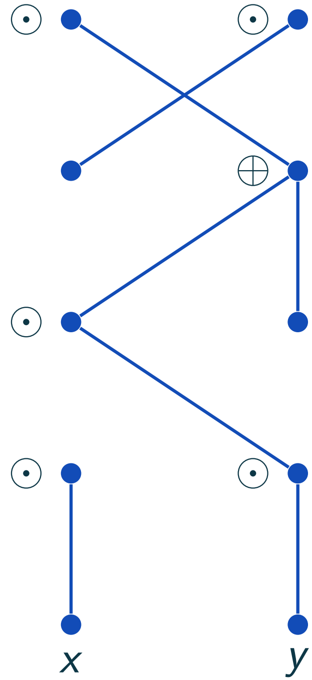

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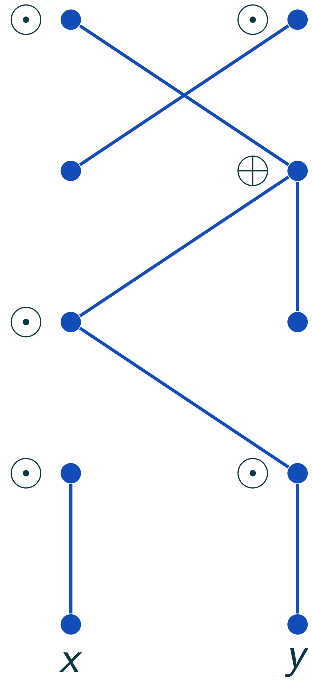

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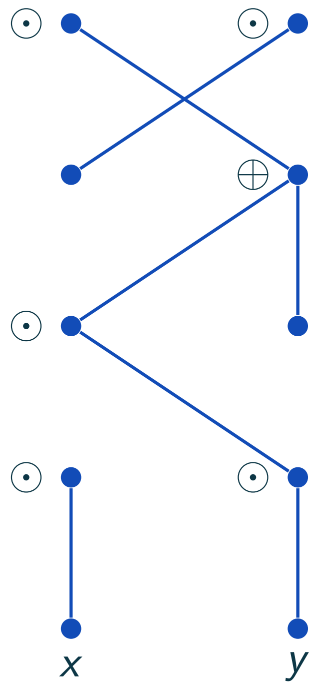
In this talk consider:

bounded alternation copyless linear CRA (BACL)



An equivalent definition of BACL

Definition

BACL is a copyless linear CRA s.t.

- Registers are ordered $x_1 < x_2 \ldots < x_k$
- x_i can only use x_i or $x_j < x_i$ in the updates
- Each registers uses always $\odot$ or always $\oplus$

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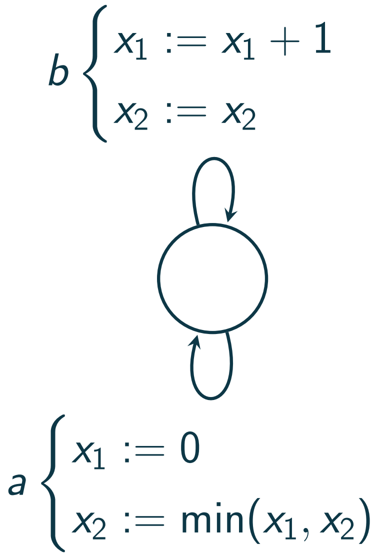
Example: "shortest block of $b's$"



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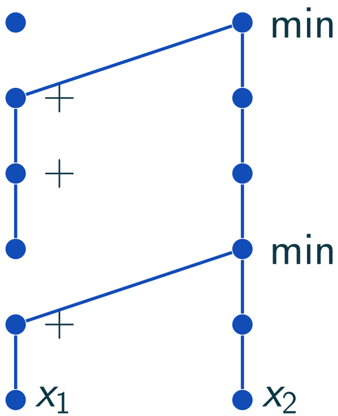
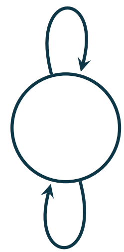


$$b \begin{cases} x_1 := x_1 + 1 \\ x_2 := x_2 \end{cases}$$



$$a \begin{cases} x_1 := 0 \\ x_2 := \min(x_1, x_2) \end{cases}$$

Was the previous encoding bounded alternation?

$$cb \begin{cases} x^{+} := x^{+} + e \\ x^{-} := x^{-} + e \\ x^{0} := x^{0} + \frac{e}{2} \\ x^{\frac{u}{2}} := x^{\frac{u}{2}} + \frac{e}{2} \\ x^{cb} := x^{cb} + e \\ x^{2cb} := x^{2cb} + 2e \end{cases}$$

$$chkcb \begin{cases} x^{cb} := 0 \\ x^{2cb} := 0 \\ x^{u} := x^{2cb} \\ x^{0} := \min(x^{0}, x^{cb}, x^{u}) \end{cases}$$

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Equivalence is:

- open for BACL over $\mathbb{N}(\min, +)$
- undecidable for BACL over $\mathbb{Z}(\min, +)$

Conclusion

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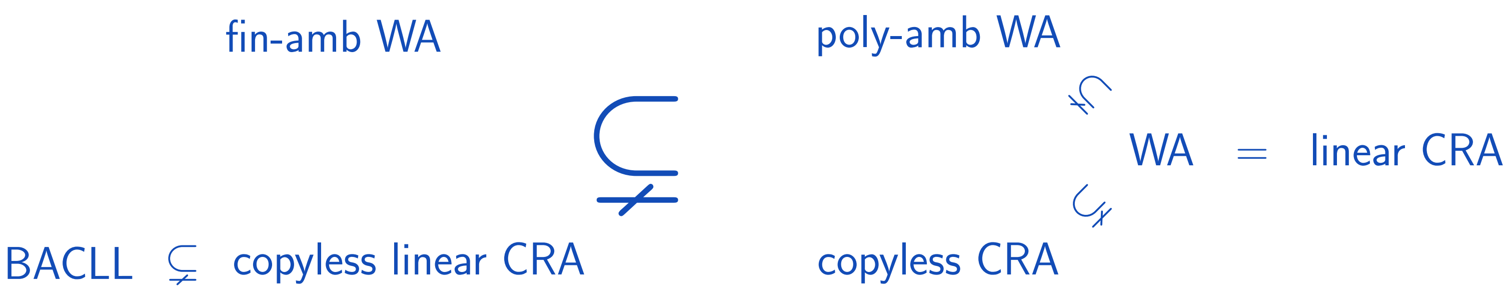
- Equivalence is undecidable for copyless linear CRA over $\mathbb{N}$
- Some hopes for the bounded alternation fragment
- Is there a nice theory to understand the picture?



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